

DECOMPOSITION OF THE RANK 3 KAC-MOODY LIE ALGEBRA  $\mathcal{F}$

WITH RESPECT TO

THE RANK 2 HYPERBOLIC SUBALGEBRA  $\mathcal{F}ib$

BY

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## Abstract

In 1983 Feingold-Frenkel studied the structure of a rank 3 hyperbolic Kac-Moody algebra  $\mathcal{F}$  containing the affine KM algebra  $A_1^{(1)}$ . In 2004 Feingold-Nicolai showed that  $\mathcal{F}$  contains all rank 2 hyperbolic KM algebras with symmetric Cartan matrices,  $A = \begin{bmatrix} 2 & -a \\ -a & 2 \end{bmatrix}$ ,  $a \geq 3$ . The case when  $a = 3$  is called  $\mathcal{Fib}$  because of its connection with the Fibonacci numbers (Feingold 1980). Some important structural results about  $\mathcal{F}$  come from the decomposition with respect to its affine subalgebra  $A_1^{(1)}$ . Here we study the decomposition of  $\mathcal{F}$  with respect to its subalgebra  $\mathcal{Fib}$ . We find that  $\mathcal{F}$  has a grading by  $\mathcal{Fib}$ -level, and prove that each graded piece,  $\mathcal{Fib}(m)$  for  $m \in \mathbb{Z}$ , is an integrable  $\mathcal{Fib}$ -module. We show that for  $|m| > 2$ ,  $\mathcal{Fib}(m)$  completely reduces as a direct sum of highest- and lowest-weight modules, and for  $|m| \leq 2$ ,  $\mathcal{Fib}(m)$  contains one irreducible non-standard quotient module  $V^{\Lambda_m} = V(m)/U(m)$ . We then prove that the quotient  $\mathcal{Fib}(m)/V(m)$  completely reduces as a direct sum of one trivial module (on level 0), and standard modules. We give an algorithm for determining the inner multiplicities of any irreducible  $\mathcal{Fib}$ -module, in particular the non-standard modules on levels  $|m| \leq 2$ . We show that multiplicities of non-standard modules on levels  $|m| = 1, 2$  do not follow the Kac-Peterson recursion (as does the non-standard adjoint representation on level 0), but instead appear to follow a recursion similar to Racah-Speiser, the recursion associated to standard modules. We also give an algorithm for finding outer multiplicities in the decompositions of all levels. We then use results of Borchers and Frenkel-Lepowsky-Meurman and construct vertex algebras from the root lattices of  $\mathcal{F}$  and  $\mathcal{Fib}$ , and study the decomposition within this setting. We find a representation  $\pi_{\mathcal{F}}$  of  $\mathcal{F}$  in  $\overline{P}_1$ , a quotient of physical-1 space by a suitable subalgebra prescribed by Borchers. We then define an action of  $\mathcal{Fib}$  on  $\overline{P}_1$  and we find extremal vectors for  $\mathcal{Fib}$  in  $\overline{P}_1$  which are not in  $\pi_{\mathcal{F}}(\mathcal{F})$ . We conjecture the existence of a “recognition algorithm” on the Schur polynomials of vectors in  $\overline{P}_1$  which will allow one to determine which vectors in  $\overline{P}_1$  are extremal with respect to  $\mathcal{Fib}$ .

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# Contents

<b>List of Tables</b> . . . . .	<b>viii</b>
<b>List of Figures</b> . . . . .	<b>x</b>
<b>Introduction</b> . . . . .	<b>1</b>
<b>1 Background</b> . . . . .	<b>4</b>
1.1 Lie algebras . . . . .	4
1.2 Kac-Moody Lie algebras . . . . .	5
1.3 The hyperbolic Kac-Moody Lie algebra $\mathcal{F}$ . . . . .	15
1.4 Feingold-Frenkel decomposition of $\mathcal{F}$ with respect to $A_1^{(1)}$ . . . . .	19
<b>2 The Fibonacci subalgebra <math>\mathcal{Fib}</math> of <math>\mathcal{F}</math></b> . . . . .	<b>23</b>
<b>3 <math>\mathcal{Fib}</math>-modules in <math>\mathcal{F}</math></b> . . . . .	<b>30</b>
3.1 The level $m$ $\mathcal{Fib}$ -module $\mathcal{Fib}(m)$ , $m \in \mathbb{Z}$ . . . . .	30
3.2 Symmetries and cosets . . . . .	33
3.3 Is $\mathcal{Fib}(m)$ completely reducible for all $m \in \mathbb{Z}$ ? . . . . .	35
3.4 Determining inner multiplicities of irreducible submodules . . . . .	41
3.5 Determining outer multiplicities of standard $\mathcal{Fib}$ -submodules . . . . .	43
<b>4 Finding decomposition data for level 0</b> . . . . .	<b>44</b>
4.1 Trivial and adjoint representations of $\mathcal{Fib}$ . . . . .	44
4.2 Highest and lowest-weight modules of $\mathcal{Fib}$ in level 0 . . . . .	45
<b>5 Finding decomposition data for levels <math>\pm 1, \pm 2</math></b> . . . . .	<b>51</b>
5.1 Inner multiplicities of the non-standard $\mathcal{Fib}$ -modules on Levels 1, 2 . . . . .	51
5.2 Outer multiplicities of $\mathcal{Fib}(\pm 1)$ and $\mathcal{Fib}(\pm 2)$ . . . . .	57
<b>6 The Vertex algebra approach</b> . . . . .	<b>60</b>
6.1 Definitions and the vertex algebra $V_{\mathcal{Fib}}$ . . . . .	60
6.2 The vertex algebra $V_{\mathcal{F}}$ . . . . .	63

6.3	Representation of $\mathcal{Fib}$ acting on $V_{\mathcal{F}}$ . . . . .	64
6.4	Finding lowest-weight vectors for $\mathcal{Fib}$ in $\overline{P}_1^{-\rho}$ . . . . .	65
<b>A</b>	<b>Kac-Peterson formulas</b> . . . . .	<b>74</b>
<b>B</b>	<b>Supplementary results</b> . . . . .	<b>75</b>
B.1	Multibracket theorem and identities used in Chapter 2 . . . . .	75
B.2	The Lie algebra representation $\pi_{\mathcal{F}} : \mathcal{F} \rightarrow \overline{P}_1$ . . . . .	77
B.3	Proving $\pi_{\mathcal{F}} _{\mathcal{F}ib} = \pi_{\mathcal{F}ib}$ . . . . .	81
<b>C</b>	<b>Dimension data for weight spaces in irreducible <math>\mathcal{Fib}</math>-modules</b> . . . . .	<b>83</b>
	<b>Bibliography</b> . . . . .	<b>102</b>

# List of Tables

3.1	Coset representative $\Lambda_m = \pi(\beta)$ for some $\beta \in \Delta_m^{re}$ , $ m  \leq 2$ . . . . .	34
3.2	A choice of real root $\beta \in \Delta_m^{re}$ for $ m  \leq 2$ and a vector $v_\beta \in \mathcal{F}_\beta$ that generates the $\mathcal{Fib}$ -module $V(m) = \mathcal{U}(\mathcal{Fib}) \cdot v_\beta$ . . . . .	40
4.1	The sequence of outer multiplicities of irreducible LW $\mathcal{Fib}$ -modules in level 0, ordered by increasing $M_0(\lambda)$ . . . . .	49
5.1	The sequence of outer multiplicities of irreducible LW $\mathcal{Fib}$ -modules in levels 1 and 2, where notation $(n_1, n_2)_m$ is as defined in Definition 5.1. . . . .	58
C1	Determination of bases of multibrackets in weight spaces $V_\mu^{\Lambda_1}$ where $\mu = n_1\beta_1 + n_2\beta_2 + \Lambda_1$ . See page 53 for explanation of notation. . . . .	84
C2	Determination of bases of multibrackets in weight spaces $V_\mu^{\Lambda_2}$ where $\mu = n_1\beta_1 + n_2\beta_2 + \Lambda_2$ . See page 53 for explanation of notation. . . . .	90
C3	Determination of bases of multibrackets in the adjoint representation $\mathcal{Fib}$ . Note: $\beta = n_1\beta_1 + n_2\beta_2$ . See page 53 for explanation of notation. . . . .	95
C4	Determination of bases of multibrackets in the module $V^{-\rho}$ on level 0. Note: $\beta = n_1\beta_1 + n_2\beta_2 - \rho$ . See page 53 for explanation of notation. . . . .	98



# List of Figures

1.1	Quadric surfaces of constant squared-length in $\mathfrak{h}_{\mathbb{R}}^*$ . . . . .	17
1.2	$P^-$ is bounded by the reflecting planes $\alpha_1^\perp, \alpha_2^\perp, \alpha_3^\perp$ . The null-cone is shown in yellow. . . . .	19
1.3	$\mathcal{Aff}(1)$ . Weights of the fundamental domain lie on the center line. . . . .	21
2.1	$\mathcal{Fib}$ -plane $\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2$ ‘slicing’ through $S_2$ , the real root hyperboloid of $\mathcal{F}$ . Shown are real roots of $\mathcal{F}$ . The two colors correspond to the two Weyl orbits of $\Phi^{re}$ . . . . .	26
2.2	The $\mathcal{Fib}$ root system $\Delta$ . Real roots lie on the red hyperbola, imaginary roots lie on the blue hyperbolas. The inner green lines show the reflecting planes $\mathbb{R}\lambda_1, \mathbb{R}\lambda_2$ . The gray lines are the $\mathcal{Fib}$ null-cone. . . . .	28
3.1	$\mathcal{Fib}$ levels -3 through 2 slicing through the hyperboloid $S_2$ . Imaginary roots on each level are not shown. Real roots lie on the blue hyperbolas $S_2 \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2 + m\gamma)$ . . . . .	31
3.2	$Q_{\mathcal{Fib}}$ is an index 5 sublattice of $P_{\mathcal{Fib}}$ in the $\mathcal{Fib}$ -plane. This figure shows only those lattice points corresponding to real or imaginary weights of $\mathcal{Fib}$ . . . . .	35
3.3	Explanation of Racah-Speiser recursion for the example of $V^{-\rho} \subset \mathcal{Fib}(0)$ . The red lines are the reflecting lines $\beta_1^\perp, \beta_2^\perp$ . . . . .	42
4.1	A comparison of the $\mathcal{F}$ - and $\mathcal{Fib}$ -multiplicities of positive roots on level 0. . .	46
4.2	A comparison of the multiplicities of negative weights of $Y(0)_-$ and $P(V^{-\rho})$ . . .	47
4.3	A comparison of the multiplicities of negative weights of $Y^1(0)_-$ and $P(V^{-2\rho})$ . . .	48
5.1	Partial weight diagrams for $\mathcal{Fib}(m), m = 1, 2$ . The multiplicities shown are $Mult_m(\mu) = \dim_{\mathcal{F}}(\mathcal{F}_\mu)$ for weights $\mu$ such that $-9 \leq wt(\mu) \leq 6$ , and are upper bounds for the inner multiplicities of the irreducible non-standard module $V^{\Lambda_m}$ . . .	51
5.2	Partial weight diagrams for non-standard $\mathcal{Fib}$ -modules $V^{\Lambda_m}$ for $m = 1$ and $2$ . The inner multiplicities shown are $Mult_{\Lambda_m}(\mu) = \dim_{V^{\Lambda_1}}(V_\mu^{\Lambda_1})$ for weights $\mu$ such that $-9 \leq wt(\mu) \leq 6$ and were calculated using the recursive algorithm presented in Section 5.1 (cf. Tables C1 and C2). . . . .	57

5.3	Partial weight diagrams for three $\mathcal{Fib}$ -modules on level 1 labeled with their multiplicities. Notation $(n_1, n_2)$ refers to weight $\Lambda_1 + n_1\beta_1 + n_2\beta_2$ , since these diagrams are all in $\mathcal{Fib}(1)$ . . . . .	58
5.4	Partial weight diagrams for three $\mathcal{Fib}$ -modules on level 2 labeled with their multiplicities. Notation $(n_1, n_2)$ refers to weight $\Lambda_2 + n_1\beta_1 + n_2\beta_2$ , since these diagrams are all in $\mathcal{Fib}(2)$ . . . . .	59

## Introduction

The theory of Kac-Moody Lie algebras has a rich history. Since their discovery in 1968 by Victor Kac and Robert Moody (independently), they have been shown to exhibit deep connections to a wide range of fields, from classical mathematics to theoretical physics. In particular, the affine Kac-Moody algebras were among the first examples to be studied, and the results in this field showed striking relationships to known combinatorial identities. Kac [K1] and Moody [M] showed that the Weyl denominator formulas for affine root systems are given by Macdonald identities. For example, the Weyl denominator formula for the affine algebra  $A_1^{(1)}$ , whose Cartan matrix is

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

is the Jacobi triple product. Throughout the late 1970s and early 1980s, further investigation into the affine algebras resulted in their explicit realization and showed more connections to power series identities. In particular, Feingold and Lepowsky [FL] showed that the values of the classical partition function are exactly the weight multiplicities in the fundamental modules for two affine algebras, one of which is  $A_1^{(1)}$ . Results such as these helped to make the class of affine algebras better understood than those of more general Kac-Moody algebras.

A natural question that arose is whether or not similar identities exist for Kac-Moody algebras of indefinite type, the easiest examples of which (in the sense of complexity) are the hyperbolic Kac-Moody algebras. Since the late 1970s a substantial amount of research has been devoted to this question, and as of yet no satisfactory closed formula for root multiplicities for the hyperbolic algebras has been found.

In 1983, the rank 3 hyperbolic extension of  $A_1^{(1)}$ , referred to by  $HA_1^{(1)}$  and  $\mathcal{F}$  in the literature and whose Cartan matrix is

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix},$$

was studied by Feingold and Frenkel [FF]. The authors constructed a subalgebra  $\mathcal{Aff} \subset \mathcal{F}$  isomorphic to  $A_1^{(1)}$ , and showed that  $\mathcal{F}$  can be expressed as a direct sum of  $\mathcal{Aff}$ -modules,

graded by level:

$$\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{A}ff(m).$$

Decomposing  $\mathcal{F}$  in this way reduced the problem to finding weight multiplicity formulas for each level with respect to  $\mathcal{A}ff$ , which for  $|m| \geq 1$  decomposed as a direct sum of irreducible integrable highest or lowest-weight (i.e., standard)  $\mathcal{A}ff$ -modules. For example, Kac had already shown that the multiplicities of roots on level 0 of  $\mathcal{F}$  with respect to  $\mathcal{A}ff$  are all 1, and the result of [FL] gave the multiplicities for the basic representation in level 1. In [FF] the authors gave formulas for level 2 which involve a ‘modified’ partition function. Subsequent works by Kang (e.g., [Ka1], [Ka2]) gave root multiplicities up to level 5.

In 2004, Feingold and Nicolai [FN] proved that every symmetric rank 2 hyperbolic KM Lie algebra  $\mathcal{H}(n)$ , whose Cartan matrices are

$$\begin{pmatrix} 2 & -n \\ -n & 2 \end{pmatrix}, \quad n \geq 3,$$

is contained in  $\mathcal{F}$ . The simplest such algebra is  $\mathcal{H}(3)$ , referred to as the ‘Fibonacci’ algebra because of its relationship to the Fibonacci numbers discovered by Feingold in 1980 [F1]. The purpose of this paper is to take an alternative but similar approach to [FF], by constructing a subalgebra of  $\mathcal{F}ib \subset \mathcal{F}$  isomorphic to  $\mathcal{H}(3)$ , and attempting to find the decomposition of  $\mathcal{F}$  with respect to  $\mathcal{F}ib$ ,

$$\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}ib(m).$$

In Chapter 1 we give background on  $\mathcal{F}$  and its rank 2 subalgebras, and review the Feingold-Frenkel decomposition with respect to  $\mathcal{A}ff$ . We also find an interesting aspect of the decomposition with respect to  $\mathcal{F}ib$  which will be investigated in more detail in Chapter 5, namely the emergence of non-standard  $\mathcal{F}ib$ -modules, that is, modules which are neither highest- nor lowest-weight.

In Chapter 2 we construct a  $\mathcal{F}ib$  subalgebra in  $\mathcal{F}$  and set up notation. In Chapter 3 we define and study the  $\mathcal{F}ib$ -modules  $\mathcal{F}ib(m)$ ,  $m \in \mathbb{Z}$ , in the  $\mathbb{Z}$ -grading of  $\mathcal{F}$ , and discuss their properties, symmetries, and weight diagrams. We analyze the decomposition of each  $\mathcal{F}ib(m)$  into irreducible  $\mathcal{F}ib$ -modules. In particular, we will show that for  $|m| > 2$ ,  $\mathcal{F}ib(m)$  completely reduces, and for  $|m| \leq 2$ ,  $\mathcal{F}ib(m)$  contains only one irreducible non-standard quotient module, one trivial module (on level 0), and standard modules.

In Chapter 4, we begin investigating the decomposition of  $\mathcal{F}$  with respect to  $\mathcal{F}ib$  by finding  $\mathcal{F}ib$ -modules on level 0. As in [FF] the adjoint representation is contained in level 0, but unlike the affine case, additional modules are found, including a trivial representation for  $\mathcal{F}ib$  and multiple copies of highest- and lowest-weight modules generated by imaginary root vectors whose roots are in the fundamental chamber for the Weyl group of  $\mathcal{F}ib$ . Using data

from Chapter 11 of [K2] and the Racah-Speiser algorithm for determining inner multiplicities of highest/lowest-weight irreducible modules, we determine all summands of the level 0 decomposition involving roots of height up to 12 (with respect to the root system of  $\mathcal{F}ib$ ).

In Chapter 5 we investigate  $\mathcal{F}ib$ -levels  $0 < |m| \leq 2$ . As a formula for determining inner multiplicities of non-standard modules is not yet known, we present an algorithm for computing these multiplicities involving finding bases of multibrackets for their weight spaces. Data for these computations are presented in tables in Appendix C. It is known that the root multiplicities of the adjoint representation of  $\mathcal{F}ib$  (which occurs as a non-standard module on level 0) obey the Kac-Peterson recursion (cf. exercises 11.11-11.12, [K2]), so it would seem reasonable to expect that the other four non-standard modules also obey a Kac-Peterson recursion. However, we will show that their multiplicities obey a recursion that appears to be of Racah-Speiser type, which indicates they have more in common with highest- and lowest-weight modules than with the adjoint.

In Chapter 6, we employ methods in [B] and [FLM] to construct a vertex algebra  $V_{\mathcal{F}}$  from the root lattice of  $\mathcal{F}$ , and a Lie algebra representation  $\pi_{\mathcal{F}} : \mathcal{F} \rightarrow \overline{P}_1$  where  $\overline{P}_1$  is a quotient of physical-1 space  $P_1$  of  $V_{\mathcal{F}}$  by a suitable subalgebra, as prescribed by Borchers. The restriction  $\pi_{\mathcal{F}}|_{\mathcal{F}ib}$  then gives a representation of  $\mathcal{F}ib$  which is compatible with the similar construction of  $V_{\mathcal{F}ib}$  from the root lattice of  $\mathcal{F}ib$ . We then investigate the decomposition of  $\mathcal{F}ib(0)$  with respect to  $\mathcal{F}ib$  within this setting by finding extremal vectors for  $\mathcal{F}ib$  in one of the weight spaces of  $\overline{P}_1$ . Although this approach will show to be more computationally intensive than that of Chapters 4 and 5, an interesting result will lead us to conjecture the existence of a “recognition algorithm” on the Schur polynomials in a given weight space of  $\overline{P}_1$  that may give extremal vectors for  $\mathcal{F}ib$ .

# Chapter 1 Background

## 1.1 Lie algebras

We start with some basic definitions and results from the theory of finite-dimensional Lie algebras (see Humphreys [H]) that will be used throughout the present work.

**Definition 1.1.** A vector space  $\mathcal{L}$  over a field  $\mathbb{F}$  with a bilinear operation  $\mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$  denoted by  $(x, y) \mapsto [x, y]$  (called the **bracket** of  $x$  and  $y$ ) is called a **Lie algebra** over  $\mathbb{F}$  if the following axioms are satisfied for all  $x, y, z \in \mathcal{L}$ :

$$[x, x] = 0 \quad \text{and} \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

As an example, consider the set of linear transformations  $End(V)$  on a vector space  $V$ . We give  $End(V)$  the structure of a Lie algebra by defining the bracket to be the commutator,  $[x, y] = x \circ y - y \circ x$ . The axioms are easily verified. This Lie algebra is called the *general linear algebra* and is denoted  $\mathfrak{gl}(V)$ . In fact, any associative algebra may be made a Lie algebra by defining the bracket in this way.

Let  $X, Y$  be subsets of a Lie algebra  $\mathcal{L}$ . The bracket  $[X, Y]$  is defined to be

$$[X, Y] = Span(\{[x, y] \mid x \in X, y \in Y\}).$$

**Definition 1.2.** Let  $\mathcal{M}$  be a vector subspace of  $\mathcal{L}$ . Then

- $\mathcal{M}$  is called an **ideal** of  $\mathcal{L}$  if  $[\mathcal{M}, \mathcal{L}] \subseteq \mathcal{M}$ , and
- $\mathcal{M}$  is called a **Lie subalgebra** of  $\mathcal{L}$  if  $[\mathcal{M}, \mathcal{M}] \subseteq \mathcal{M}$ .

If  $\mathcal{J}$  is an ideal of  $\mathcal{L}$ , then the quotient vector space  $\mathcal{L}/\mathcal{J}$  is also a Lie algebra with bracket defined by  $[\bar{x}, \bar{y}] = \overline{[x, y]}$  for all  $x, y \in \mathcal{L}$ , where  $\bar{x} = x + \mathcal{J} \in \mathcal{L}/\mathcal{J}$ .

**Definition 1.3.** The **universal enveloping algebra**  $\mathcal{U}(\mathcal{L})$  of a Lie algebra  $\mathcal{L}$  is the associative algebra  $\mathcal{U}(\mathcal{L}) = \mathcal{T}(\mathcal{L})/\mathcal{I}$  where  $\mathcal{T}(\mathcal{L}) = \mathbb{F} \oplus \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus (\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}) \oplus \dots$  is the tensor algebra of the underlying vector space of  $\mathcal{L}$ , and  $\mathcal{I}$  is the ideal of  $\mathcal{T}(\mathcal{L})$  generated by elements of the form  $x \otimes y - y \otimes x - [x, y]$  for  $x, y \in \mathcal{L}$ .

**Definition 1.4.** Given Lie algebras  $\mathcal{L}$  and  $\mathcal{M}$  over  $\mathbb{F}$ , a linear transformation  $\phi : \mathcal{L} \rightarrow \mathcal{M}$  is called a **Lie algebra homomorphism** if  $\phi([x, y]) = [\phi(x), \phi(y)]$  for all  $x, y \in \mathcal{L}$ .

**Definition 1.5.** A **representation** of a Lie algebra  $\mathcal{L}$  is a homomorphism  $\phi : \mathcal{L} \rightarrow \mathfrak{gl}(V)$  for some vector space  $V$  over  $\mathbb{F}$ .

Sometimes  $V$  itself is referred to as the representation of  $\mathcal{L}$  in the literature.

**Definition 1.6.** The **adjoint representation** of a Lie algebra  $\mathcal{L}$  is the representation  $ad : \mathcal{L} \rightarrow \mathfrak{gl}(\mathcal{L})$  defined by  $ad_x(y) = [x, y]$ .

**Definition 1.7.** Given a Lie algebra  $\mathcal{L}$ , a vector space  $V$  is called an  **$\mathcal{L}$ -module** if there is a bilinear action  $\mathcal{L} \times V \rightarrow V$ , denoted  $(x, v) \mapsto x \cdot v$ , such that  $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$  for  $x, y \in \mathcal{L}$  and  $v \in V$ . Also, a subspace  $W$  of  $V$  is called an  **$\mathcal{L}$ -submodule of  $V$**  if  $W$  is itself an  $\mathcal{L}$ -module, that is,  $\mathcal{L} \cdot W \subseteq W$ .

Having a representation  $\phi : \mathcal{L} \rightarrow \mathfrak{gl}(V)$  is equivalent to  $V$  being an  $\mathcal{L}$ -module, by  $x \cdot v = \phi(x)v$ .

**Definition 1.8.** An  $\mathcal{L}$ -module  $V$  is called **irreducible** if its only submodules are the trivial module,  $\{0\}$ , and  $V$  itself.

**Definition 1.9.** Given a collection of  $\mathcal{L}$ -modules  $\{V_i\}_{i \in I}$  where  $I$  is an index set, then the **vector space direct sum**  $\bigoplus_{i \in I} V_i$  is also an  $\mathcal{L}$ -module, where the action is component-wise.

**Definition 1.10.** A module is called **completely reducible** if it is a direct sum of irreducible modules.

## 1.2 Kac-Moody Lie algebras

Kac and Moody generalized the theory of finite-dimensional Lie algebras to include certain other classes of infinite-dimensional Lie algebras, which will be described shortly.

**Definition 1.11.** An  $\ell \times \ell$  integer matrix  $A = (a_{ij})$  is called a **generalized Cartan matrix (GCM)** if it has the the following properties:

$$a_{ii} = 2 \text{ for } 1 \leq i \leq \ell, \quad a_{ij} \leq 0 \text{ if } i \neq j, \quad \text{and } a_{ij} = 0 \text{ implies } a_{ji} = 0.$$

A GCM  $A$  is called **symmetrizable** if there exists a diagonal matrix  $D = \text{diag}(d_1, \dots, d_\ell)$ ,  $d_i > 0$ , and  $A = DS$  for  $S = (s_{ij})$  symmetric.

$A$  is called **indecomposable** if it cannot be put into block-diagonal form  $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ , where each  $A_i$  is non-trivial, through a relabeling of the rows and columns.

A symmetrizable GCM  $A = DS$  is of:

- **finite type** if  $S$  is positive definite.

- **affine type** if  $S$  is positive semi-definite and has corank 1.
- **indefinite type** if  $S$  is indefinite. In addition, if every submatrix of  $A$  is of either finite or affine type, then  $A$  is of **hyperbolic type**.

From now on we assume all GCMs to be symmetrizable.

**Definition 1.12.** Given a GCM  $A$ , a **realization of  $A$  over  $\mathbb{F}$**  is a triple  $(\mathfrak{h}, \Pi, \Pi^\vee)$  where  $\mathfrak{h}$  is a complex vector space,  $\Pi = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{h}^*$  (the set of **simple roots**), and  $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subset \mathfrak{h}$  (the set of **simple coroots**), satisfying the conditions

- $\Pi$  and  $\Pi^\vee$  are linearly independent,
- $\alpha_i(\alpha_j^\vee) = a_{ji}$  for  $1 \leq i, j \leq \ell$ ,
- $\ell - \text{rank}(A) = \dim \mathfrak{h} - \ell$ .

**Definition 1.13.** Let  $A$  be an  $\ell \times \ell$  GCM with realization  $(\mathfrak{h}, \Pi, \Pi^\vee)$ . The **Kac-Moody Lie algebra**, or **KM algebra**,  $\mathcal{L} = \mathcal{L}(A)$  over  $\mathbb{F}$  associated to  $A$  is the Lie algebra generated by  $\{e_i, f_i \mid i = 1 \dots \ell\}$  (called the **Chevalley generators**) and  $\mathfrak{h}$ , subject to the **Serre relations**,

- $[h, h'] = 0$  for all  $h, h' \in \mathfrak{h}$ ,
- $[h, e_i] = \alpha_i(h)e_i$  and  $[h, f_i] = -\alpha_i(h)f_i$ , for all  $h \in \mathfrak{h}^*$ ,
- $[e_i, f_j] = \delta_{ij}\alpha_i^\vee$ ,
- $(ad_{e_i})^{-a_{ij}+1}(e_j) = 0, \ i \neq j$  and  $(ad_{f_i})^{-a_{ij}+1}(f_j) = 0, \ i \neq j$ .

The abelian subalgebra  $\mathfrak{h}$  of  $\mathcal{L}$  is called the **Cartan subalgebra**.

Under the adjoint action,  $ad_h : \mathcal{L} \rightarrow \mathcal{L}$  given by  $ad_h(x) = [h, x]$ ,  $\mathfrak{h}$  acts simultaneously diagonalizably on  $\mathcal{L}$ . The simultaneous eigenspaces

$$\mathcal{L}_\alpha = \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x, \ h \in \mathfrak{h}\}$$

are labelled by certain linear functionals  $\alpha \in \mathfrak{h}^*$ . In particular, for  $1 \leq i \leq \ell$ ,

$$\mathcal{L}_0 = \mathfrak{h}, \quad \mathcal{L}_{\alpha_i} = \mathbb{F}e_i \quad \text{and} \quad \mathcal{L}_{-\alpha_i} = \mathbb{F}f_i.$$

**Definition 1.14.** We call  $\alpha \in \mathfrak{h}^*$  a **root** when  $\alpha \neq 0$  and  $\mathcal{L}_\alpha \neq 0$ , in which case  $\mathcal{L}_\alpha$  is called the  $\alpha$  **root space**, and the **multiplicity** of root  $\alpha$  is

$$\text{Mult}_{\mathcal{L}}(\alpha) = \dim(\mathcal{L}_\alpha).$$

The **root lattice** is

$$Q = \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*.$$

The set of all roots is denoted by  $\Delta$  (so  $\Delta \subset Q$ ).



For roots  $\alpha, \beta \in \Delta \cup \{0\}$ , we have that  $[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subseteq \mathcal{L}_{\alpha+\beta}$ .

**Definition 1.15.** The *Cartan involution* of  $\mathcal{L}$  is an order-two automorphism  $\nu : \mathcal{L} \rightarrow \mathcal{L}$  determined by  $\nu(e_i) = -f_i$ ,  $\nu(f_i) = -e_i$  for  $1 \leq i \leq \ell$ , and  $\nu(h) = -h$  for  $h \in \mathfrak{h}$ .

For every  $\alpha \in \Delta$ , we have that  $\nu(\mathcal{L}_\alpha) = \mathcal{L}_{-\alpha}$ , thus  $\text{Mult}(\alpha) = \text{Mult}(-\alpha)$ . We also let  $\nu(\alpha) = -\alpha$ .

The set of roots  $\Delta$  can be partitioned into “positive” and “negative” roots,

$$\Delta = \Delta_+ \cup \Delta_-,$$

where

$$\Delta_+ = \left\{ \sum_{i=1}^{\ell} n_i \alpha_i \in \Delta \mid 0 \leq n_i \in \mathbb{Z} \right\}$$

and  $\Delta_- = -\Delta_+$ . There is a partial order on  $\Delta$  defined by

$$\mu \leq \lambda \text{ if and only if } \lambda - \mu = \sum_{i=1}^{\ell} k_i \alpha_i \text{ where } k_i \in \mathbb{Z}_{\geq 0}. \quad (1.1)$$

**Definition 1.16.** The *height* of a root  $\alpha = \sum_{i=1}^{\ell} n_i \alpha_i \in \Delta$  is  $ht(\alpha) = \sum_{i=1}^{\ell} n_i$ .

Let  $\mathfrak{n}_+$  (resp.  $\mathfrak{n}_-$ ) be the subalgebra generated by  $\{e_i \mid 1 \leq i \leq \ell\}$  (resp.  $\{f_i \mid 1 \leq i \leq \ell\}$ ). Then

$$\mathfrak{n}_{\pm} = \bigoplus_{\alpha \in \Delta_{\pm}} \mathcal{L}_{\pm\alpha}$$

and by Theorem 1.2 of [K2],  $\mathcal{L}$  has the triangular decomposition and root space decomposition,

$$\mathcal{L} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \quad \text{and} \quad \mathcal{L} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathcal{L}_{\alpha},$$

respectively.

We also have the following fundamental results of Kac.

**Theorem 1.17** ([K2]). Let  $\mathcal{L} = \mathcal{L}(A)$  be the KM algebra associated to a symmetrizable GCM  $A$ . Then

- there exists a non-degenerate symmetric bilinear form  $(\cdot, \cdot)$  on  $\mathcal{L}$  that is invariant (i.e.,  $([x, y], z) = (x, [y, z])$  for all  $x, y, z \in \mathcal{L}$ ), and
- the restriction  $(\cdot, \cdot)|_{\mathfrak{h}}$  is non-degenerate, giving an induced non-degenerate bilinear form  $(\cdot, \cdot)$  on  $\mathfrak{h}^*$  such that  $(\alpha_i, \alpha_j) = s_{ij}$  and  $d_i = \frac{2}{(\alpha_i, \alpha_i)}$  for  $1 \leq i, j \leq \ell$ .

From now on,  $(\cdot, \cdot)$  shall refer to the induced bilinear form on  $\mathfrak{h}^*$ .

**Definition 1.18.** For  $\alpha, \beta \in \Delta$ , the *angle bracket* of  $\alpha, \beta$  is given by

$$\langle \alpha, \beta \rangle := \frac{2(\alpha, \beta)}{(\alpha, \alpha)}.$$

Note that  $\langle \cdot, \cdot \rangle$  is linear only in the second, and that  $\langle \beta, \alpha \rangle = \frac{(\alpha, \alpha)}{(\beta, \beta)} \langle \alpha, \beta \rangle$ . Also,  $\langle \alpha_i, \alpha_j \rangle = d_i s_{ij} = a_{ij} = \alpha_j(\alpha_i^\vee)$ , thus

$$A = \left( \langle \alpha_i, \alpha_j \rangle \right)_{1 \leq i, j \leq \ell}.$$

The integral entries  $\langle \alpha_i, \alpha_j \rangle$  ( $1 \leq i, j \leq \ell$ ) of the GCM are referred to as *Cartan integers*. Also since the angle bracket is linear in the second, we have that  $\langle \alpha_i, \alpha \rangle = \alpha(\alpha_i^\vee)$  for  $\alpha \in \mathfrak{h}^*$ .

**Definition 1.19.** The **squared-length** of a root  $\alpha$  is  $\|\alpha\|^2 = (\alpha, \alpha)$ .

For indefinite KM algebras, it is possible for roots to have negative squared-length.

For each  $1 \leq i \leq \ell$ , let

$$h_i = \frac{\alpha_i^\vee}{d_i} = \frac{(\alpha_i, \alpha_i)}{2} \alpha_i^\vee \in \mathfrak{h}.$$

**Remark 1.20.** Note that if  $A$  is symmetric, then  $(\alpha_i, \alpha_j) = a_{ij}$ ,  $\|\alpha_i\|^2 = 2$ , and  $h_i = \alpha_i^\vee$ , giving us  $\alpha_i(h_j) = \alpha_i(\alpha_j^\vee) = a_{ji} = a_{ij}$ .

**Definition 1.21.** Given a KM algebra  $\mathcal{L}$  and its associated simple roots  $\Pi$ , the **fundamental weights** of  $\mathcal{L}$ ,  $\omega_j \in \mathfrak{h}^*$  for  $1 \leq j \leq \ell$ , are defined by

$$\langle \alpha_i, \omega_j \rangle = \omega_j(\alpha_i^\vee) = \delta_{ij} \quad \text{or, equivalently,} \quad \langle \omega_i, \alpha_j \rangle = \frac{(\alpha_i, \alpha_i)}{(\omega_i, \omega_i)} \delta_{ij}.$$

The **weight lattice**  $P$  of  $\mathcal{L}$  is the set of **integral weights**,

$$P = \sum_{i=1}^{\ell} \mathbb{Z} \omega_i,$$

and the **dominant integral weights** are defined as

$$P^+ = \sum_{i=1}^{\ell} \mathbb{Z}^+ \omega_i = \left\{ \sum_{i=1}^{\ell} n_i \omega_i \mid 0 \leq n_i \in \mathbb{Z} \right\} = \{ \omega \in P \mid \langle \alpha_j, \omega \rangle \geq 0, 1 \leq j \leq \ell \}.$$

Writing a simple root  $\alpha_j$  as a linear combination of the fundamental weights,

$$\alpha_j = \sum_{k=1}^{\ell} c_{kj} \omega_k, \quad \Rightarrow \quad a_{ij} = \langle \alpha_i, \alpha_j \rangle = \sum_{k=1}^{\ell} c_{kj} \langle \alpha_i, \omega_k \rangle = \sum_{k=1}^{\ell} c_{kj} \delta_{ik} = c_{ij}.$$

Hence,

$$\alpha_j = \sum_{i=1}^{\ell} a_{ij} \omega_i.$$

Thus, the  $j$ -th column of the GCM is the coordinate vector of  $\alpha_j$  with respect to the basis of the fundamental weights. If  $A$  is invertible, the columns of the inverse of the GCM give the coefficients of the fundamental weights in the basis of the simple roots:

$$\omega_j = \sum_{i=1}^{\ell} (A^{-1})_{ij} \alpha_i.$$

Moreover, since  $\langle \omega_i, \alpha_j \rangle = \frac{(\alpha_i, \alpha_j)}{(\omega_i, \omega_i)} \delta_{ij}$  we have the following:

$$\langle \omega_i, \omega_j \rangle = \langle \omega_i, \sum_{i=1}^{\ell} (A^{-1})_{ij} \alpha_i \rangle = \frac{(\alpha_i, \alpha_i)}{(\omega_i, \omega_i)} (A^{-1})_{ij}.$$

Hence the inverse of the Cartan matrix is:

$$(A^{-1})_{1 \leq i, j \leq \ell} = \text{diag} \left( \frac{(\omega_i, \omega_i)}{(\alpha_i, \alpha_i)} \right)_{1 \leq i \leq \ell} \left( \langle \omega_i, \omega_j \rangle \right)_{1 \leq i, j \leq \ell}$$

**Definition 1.22.** We define the function  $wt : P \rightarrow \mathbb{Z}$  by  $wt(\omega) = \sum_{i=1}^{\ell} n_i$  if  $\omega = \sum_{i=1}^{\ell} n_i \omega_i \in P$ . We call  $\omega$  a **positive weight** if  $wt(\omega) > 0$  and a **negative weight** if  $wt(\omega) < 0$ .

**Definition 1.23.** The **Weyl group**  $W = W(A)$  associated to  $\mathcal{L}(A)$  is the group generated by **simple reflections**  $\{w_i \mid i = 1 \leq i \leq \ell\}$ , where for all  $\lambda \in \mathfrak{h}^*$ ,

$$w_i \lambda = \lambda - \langle \alpha_i, \lambda \rangle \alpha_i.$$

Note that  $w_i$  fixes the hyperplane  $\alpha_i^\perp = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda, \alpha_i) = 0\}$  pointwise and  $w_i \alpha_i = -\alpha_i$ . Thus, each  $w_i$  is a reflection, so  $W$  is a group of orthogonal transformations with respect to  $(\cdot, \cdot)$ . There is a group homomorphism  $sgn : W \rightarrow \{\pm 1\}$  defined by  $sgn(w_i) = -1$ .

Furthermore, Kac gives us:

**Lemma 1.24** ([K3]). Let  $\mathcal{L}$  be a KM algebra with Weyl group  $W$ , set of roots  $\Delta$ , simple roots  $\Pi$ , and root lattice  $Q$ .

- a) The set of roots  $\Delta$  is  $W$ -invariant, and  $Mult(\alpha) = Mult(w(\alpha))$  for every  $w \in W$  and  $\alpha \in \Delta$ . The set  $\Delta_+ \setminus \{\alpha_i\}$  is invariant with respect to  $w_i$ .
- b) The set  $\Delta_+$  is uniquely defined by the following properties:
  - i)  $\Pi \subset \Delta_+ \subset Q$ ;  $2\alpha \notin \Delta_+$  if  $\alpha \in \Pi$ ,
  - ii) If  $\alpha \in \Delta_+ \setminus \Pi$ , then  $\alpha + k\alpha_i \in \Delta_+$  if and only if  $-p \leq k \leq q, k \in \mathbb{Z}$ , where  $p, q$  are some non-negative integers satisfying  $p - q = \langle \alpha_i, \alpha \rangle$ .
- c) If  $A$  is indecomposable of affine or hyperbolic type, then if  $\alpha \in \Delta_+$ , there exists  $\beta \in \Pi$  such that  $\alpha + \beta \in \Delta_+$ .

If  $A$  is a finite-type Cartan matrix, then  $W$  and  $\Delta$  are both finite,  $(\cdot, \cdot)$  is positive definite, and  $W\Pi = \Delta$ . Also,  $\Delta$  embeds into Euclidean space  $\mathbb{R}^\ell$ , and satisfies the properties of a finite root system [H].

If  $A$  is a GCM we call  $\Delta$  a **generalized root system**, and it partitions into *real roots* and *imaginary roots*,

$$\Delta = \Delta^{re} \cup \Delta^{im}, \quad \text{where} \quad \Delta^{re} = W\Pi \quad \text{and} \quad \Delta^{im} = \Delta \setminus \Delta^{re}.$$

**Definition 1.25.** Let  $V$  be an  $\mathcal{L}$ -module corresponding to representation  $\phi : \mathcal{L} \rightarrow \text{End}(V)$ . We say that  $x \in \mathcal{L}$  is **locally nilpotent** on  $V$  if for all  $v \in V$ , there exists a positive integer  $n$  such that  $\phi(x)^n v = 0$ . We sometimes write more briefly  $x^n \cdot v$  for  $\phi(x)^n v$  with an abuse of notation.

**Definition 1.26.** For  $\lambda \in \mathfrak{h}^*$ , the  $\lambda$  **weight space**  $V_\lambda$  of an  $\mathcal{L}$ -module  $V$  is

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda(h)v \text{ for } h \in \mathfrak{h}\},$$

and  $\lambda$  is a **weight** of  $V$  if  $V_\lambda \neq 0$ .

**Definition 1.27.** An  $\mathcal{L}$ -module  $V$  is called  **$\mathfrak{h}$ -diagonalizable** if  $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ .

Given an  $\mathfrak{h}$ -diagonalizable  $\mathcal{L}$ -module  $V$ , let

$$P(V) = \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0\} \quad \text{and} \quad D(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda\}.$$

**Definition 1.28.** The **category  $\mathcal{O}$**  is the category whose objects are  $\mathfrak{h}$ -diagonalizable  $\mathcal{L}$ -modules  $V$  with finite-dimensional weight spaces and such that for each  $V$ , there exists a finite number of elements  $\lambda_1, \dots, \lambda_s \in \mathfrak{h}^*$  such that

$$P(V) \subset \bigcup_{i=1}^s D(\lambda_i).$$

The morphisms of the category  $\mathcal{O}$  are  $\mathcal{L}$ -module homomorphisms.

**Definition 1.29.** An  $\mathcal{L}$ -module  $V$  is a **highest-weight module** with **highest weight**  $\lambda \in \mathfrak{h}^*$  if there exists a non-zero vector  $v_\lambda \in V$ , called a **highest-weight vector (HWW)**, with the following properties:

$$\mathfrak{n}_+ \cdot v_\lambda = 0, \quad h \cdot v_\lambda = \lambda(h)v_\lambda \text{ for all } h \in \mathfrak{h}^*, \quad \text{and} \quad \mathcal{U}(\mathfrak{n}_-) \cdot v_\lambda = V.$$

If  $V$  is irreducible, we write  $V = V^\lambda$ . The  $\mu$  weight space of  $V^\lambda$  is denoted  $V_\mu^\lambda$ . In particular, note that the second property implies  $v_\lambda \in V_\lambda^\lambda$ .

A highest-weight  $\mathcal{L}$ -module is in the category  $\mathcal{O}$ .

**Definition 1.30.** Let  $V$  be a completely reducible  $\mathcal{L}$ -module in the category  $\mathcal{O}$ , and let  $\mu \in P(V)$ . Define

$$\text{High}_V(\mu) = \{v \in V_\mu \mid e_i \cdot v = 0, \ 1 \leq i \leq \ell\}$$

to be the subspace of highest-weight vectors in  $V_\mu$ . Define

$$M_V(\mu) = \dim \text{High}_V(\mu)$$

to be the **outer multiplicity of  $\mu$  in  $V$** , and define

$$P'(V) = \{\mu \in P(V) \mid M_V(\mu) > 0\}$$

to be the set of weights whose weight spaces of  $V$  contain highest-weight vectors.

Let  $\mathcal{B}(\mu) = \{v_{\mu,i} \in V_\mu \mid 1 \leq i \leq M_V(\mu)\}$  be a basis for  $\text{High}_V(\mu)$ . Then each  $v_{\mu,i}$  generates an irreducible highest-weight  $\mathcal{L}$ -module,  $\mathcal{U}(\mathcal{L}) \cdot v_{\mu,i} \simeq V^\mu$ .

**Definition 1.31.** An  $\mathfrak{h}$ -diagonalizable module  $V$  over  $\mathcal{L}$  is **integrable** if the generators  $\{e_i, f_i \mid 1 \leq i \leq \ell\}$  of  $\mathcal{L}$  are all locally nilpotent on  $V$ .

**Lemma 1.32** ([K2] 10.1). Let  $\mathcal{L}$  be a KM algebra and  $V^\lambda$  be an irreducible highest-weight  $\mathcal{L}$ -module. Then  $V^\lambda$  is integrable if and only if  $\lambda \in P^+$ .

By the lemma, we have for any integrable highest-weight  $\mathcal{L}$ -module  $V$ ,

$$P'(V) \subset P^+. \quad (1.2)$$

**Definition 1.33.** For fixed  $\alpha \in \Delta$  and  $\lambda \in P$ , the set  $S_\alpha(\lambda) = \{\lambda - i\alpha \mid 0 \leq i \leq \langle \alpha, \lambda \rangle\}$  is called the  $\alpha$ -weight string through  $\lambda$ . A subset  $R \subset P$  of integral weights is called **saturated** if for all  $\lambda \in R$ ,  $\alpha \in \Delta$ ,  $S_\alpha(\lambda) \subset R$ .

Observe that weight strings and saturated sets of weights are  $W$ -invariant. If  $V$  is an  $\mathcal{L}$ -module, then  $P(V)$  is a saturated set since

$$\bigcup_{i=1}^{\ell} S_{\alpha_i}(\mu) \subset P(V) \quad \text{for all } \mu \in P(V) \quad (1.3)$$

**Theorem 1.34** ([K2] 10.7). Let  $\mathcal{L}$  be a KM algebra and let  $V$  be an  $\mathcal{L}$ -module in the category  $\mathcal{O}$ . Then  $V$  is integrable if and only if  $V$  is completely reducible, that is,  $V$  has a decomposition

$$V = \bigoplus_{\lambda \in P'(V)} M_V(\lambda) V^\lambda.$$

**Definition 1.35.** Let  $V$  be an  $\mathcal{L}$ -module from the category  $\mathcal{O}$ . The **formal character** of  $V$  is defined as

$$\text{Ch}(V) = \sum_{\lambda \in P(V)} \dim V_\lambda e^\lambda,$$

where  $e^\lambda$  is a formal exponential satisfying  $e^\lambda e^\mu = e^{\lambda+\mu}$ .

If  $V$  is an integrable highest-weight module and  $\mu \in P(V)$ , then define

$$\text{Mult}_V(\mu) = \dim_V(V_\mu). \quad (1.4)$$

**Proposition 1.36** ([K2] 10.1). If  $V = V^\lambda$  is an irreducible highest-weight  $\mathcal{L}$ -module with highest weight  $\lambda \in P^+$ , then for all  $\mu \in P(V)$  and  $w \in W$ ,

$$\text{Mult}_V(\mu) = \text{Mult}_V(w\mu).$$

In particular,  $P(V)$  is  $W$ -invariant.

**Theorem 1.37** ([K2] 10.4). *Let  $V^\lambda$  be an irreducible  $\mathcal{L}$ -module with highest weight  $\lambda \in P^+$ , and let  $\rho = \sum_{i=1}^{\ell} \omega_i$ . Then*

$$Ch(V^\lambda) = \frac{\sum_{w \in W} sgn(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{Mult(\alpha)}}.$$

*This is known as the **Weyl-Kac character formula**.*

Racah and Speiser gave a recursive algorithm based on the Weyl-Kac character formula (the so-called “Racah-Speiser recursion”) for determining weight multiplicities of irreducible highest-weight modules for KM algebras. Section 3.4 gives a detailed description of how this recursion is used in the setting of  $\mathcal{F}ib$ -modules in  $\mathcal{F}$ . Kac and Peterson gave a separate algorithm for determining multiplicities of weights in the adjoint representation (roots) of a KM algebra [K2]. This “Kac-Peterson recursion” is outlined briefly in Appendix A.

**Definition 1.38.** *Given a GCM  $A$  and its associated Kac-Moody Lie algebra  $\mathcal{L}(A)$ , we define the shorthand notation  $e_{i_n \dots i_2 i_1}$  to stand for the multibracket  $ad_{e_{i_n}} \dots ad_{e_{i_2}} e_{i_1}$ , which lies in the root space  $\mathcal{L}_\alpha$  for  $\alpha = \sum_{j=1}^n \alpha_{i_j} \in \Delta_+$ . Similarly, we write  $f_{i_n \dots i_2 i_1}$  as shorthand for the multibracket  $ad_{f_{i_n}} \dots ad_{f_{i_2}} f_{i_1} \in \mathcal{L}_{-\alpha}$ .*

For a fixed  $\alpha \in \Delta_+$ , the set of multibrackets  $e_{i_n \dots i_2 i_1} \in \mathcal{L}_\alpha$  (where by definition,  $\sum_{j=1}^n \alpha_{i_j} = \alpha$ ) spans  $\mathcal{L}_\alpha$ , and the set of multibrackets  $f_{i_n \dots i_2 i_1} \in \mathcal{L}_{-\alpha}$  spans  $\mathcal{L}_{-\alpha}$ . Furthermore, if  $e_{i_n \dots i_2 i_1} \neq 0$ , then for all  $1 \leq k \leq n$ ,  $e_{i_k \dots i_2 i_1} \neq 0$ . In other words, each partial sum  $\sum_{j=1}^k \alpha_{i_j}$  is also a root in  $\Delta_+$ . Similarly, if  $f_{i_n \dots i_2 i_1} \neq 0$ , then for all  $1 \leq k \leq n$ ,  $f_{i_k \dots i_2 i_1} \neq 0$  and  $-\sum_{j=1}^k \alpha_{i_j}$  is a root in  $\Delta_-$ .

We now briefly outline how one may determine weight multiplicities of an irreducible highest-weight  $\mathcal{L}$ -module  $V^\lambda$  by recursively computing bases for its weight spaces. (A detailed description of this process for irreducible  $\mathcal{F}ib$ -modules is given in Chapter 5.) First,  $V_\lambda^\lambda$  has basis consisting of a single highest-weight vector  $\{v_\lambda\}$ . Let  $\lambda \neq \mu \in P(\lambda)$  and assume for each  $1 \leq i \leq \ell$  we have the previously determined basis  $\mathcal{B}_i$  for  $V_{\mu+\alpha_i}^\lambda$ . Then acting with  $f_i$  on each vector in  $\mathcal{B}_i$  for  $1 \leq i \leq \ell$  gives a spanning set  $\mathcal{S}_\mu = \{v_1, \dots, v_n\}$  for  $V_\mu^\lambda$ , where  $n = \sum_{i=1}^{\ell} |\mathcal{B}_i|$ . Linear dependence relations on the vectors in  $\mathcal{S}_\mu$  are then found by solving the homogeneous system of linear equations determined by setting

$$e_i \cdot \sum_{j=1}^n c_j v_j = \sum_{j=1}^n c_j (e_i \cdot v_j) = 0$$

for  $1 \leq i \leq \ell$ . Since  $V^\lambda$  is an irreducible module and therefore cannot contain any more highest-weight vectors, any nontrivial solutions will yield dependence relations on the vectors in  $\mathcal{S}_\mu$ . Then choose vectors to delete from the spanning set to obtain a basis  $\mathcal{B}(\mu)$  for  $V_\mu^\lambda$ . Finally, we have  $Mult_\lambda(\mu) = |\mathcal{B}(\mu)|$ .

**Definition 1.39.** An  $\mathcal{L}$ -module  $V$  is a **lowest-weight module** with **lowest weight**  $\lambda \in \mathfrak{h}^*$  if there exists a non-zero vector  $v_\lambda \in V$ , called a **lowest-weight vector (LWV)**, with the following properties:

$$\mathfrak{n}_- \cdot v_\lambda = 0, \quad h \cdot v_\lambda = \lambda(h)v_\lambda, \quad \text{and} \quad \mathcal{U}(\mathfrak{n}_+) \cdot v_\lambda = V.$$

If  $V$  is irreducible we write  $V = V^\lambda$  and denote the  $\mu$ -weight space of  $V$  by  $V_\mu^\lambda$ .

Given an irreducible integrable  $\mathfrak{h}$ -diagonalizable highest-weight  $\mathcal{L}$ -module  $V^\lambda$  in  $\mathcal{O}$  with HWV  $v_\lambda$ , we may view the vector space  $V^\lambda$  as a different  $\mathcal{L}$ -module in the opposite category  $\mathcal{O}^{op}$  under the action of  $\mathcal{L}$  twisted by the Cartan involution  $\nu$ . This new module is denoted  $V^{-\lambda}$  and has set of weights  $P(V^{-\lambda}) = -P(V^\lambda)$ . Define the new action  $\circ$  of  $\mathcal{L}$  on  $V^\lambda$ ,

$$x \circ v := \nu(x) \cdot v \quad \text{for } x \in \mathcal{L}, v \in V,$$

so that

$$\begin{aligned} [x, y] \circ v &= \nu[x, y] \cdot v = [\nu(x), \nu(y)] \cdot v = \nu(x) \cdot (\nu(y) \cdot v) - \nu(y) \cdot (\nu(x) \cdot v) \\ &= x \circ (y \circ v) - y \circ (x \circ v). \end{aligned}$$

Then if  $v_\mu \in V_\mu^\lambda$  where  $\mu \in P(V^\lambda)$ , we have

$$h \circ v_\mu = -h \cdot v_\mu = -\mu(h)v_\mu \quad \text{and} \quad f_i \circ v_\lambda = -e_i \cdot v_\lambda = 0 \quad \text{for } 1 \leq i \leq \ell.$$

Then  $V_{-\mu}^{-\lambda}$  is the weight space  $V_\mu^\lambda$  viewed under the  $\circ$  action as the  $-\mu$ -weight space of  $V^{-\lambda}$ . Thus the module  $V^{-\lambda} = \bigoplus_{\lambda \in -P(V^\lambda)} V_{-\mu}^{-\lambda}$  constructed in this way is called the *contragredient module* to  $V^\lambda$ , and this construction of a lowest-weight module from a given highest-weight module is equivalent to a functor relating  $\mathcal{O}$  and  $\mathcal{O}^{op}$ .

**Remark 1.40.** For every statement on category  $\mathcal{O}$  modules previously mentioned, an analogous statement also holds for modules in category  $\mathcal{O}^{op}$ . In particular, we have

1. (Definition 1.30) For  $V$  a not necessarily irreducible  $\mathcal{L}$ -module in category  $\mathcal{O}^{op}$  and  $\mu \in P(V)$ , define  $\text{Low}_V(\mu) = \{v \in V_\mu \mid f_i \cdot v = 0, 1 \leq i \leq \ell\}$  to be the subspace of lowest-weight vectors in  $V_\mu$ . Then the outer multiplicity  $M_V(\mu) = \dim \text{Low}_V(\mu)$ . If  $V = V^\lambda$  is irreducible, then  $M_{V^\lambda}(\mu) = M_{V^{-\lambda}}(-\mu)$ .
2. (Lemma 1.32) If  $V^\lambda$  is an irreducible lowest-weight module, then  $V^\lambda$  is integrable if and only if  $\lambda \in P^-$ .
3. (Theorem 1.34) If  $V$  is a module in category  $\mathcal{O}^{op}$ , then  $V$  is integrable if and only if  $V$  is completely reducible, so that  $V = \bigoplus_{\lambda \in P'(V)} M_V(\lambda)V^\lambda$ , where  $P'(V) \subset P^-$  is the set of weights whose weight spaces of  $V$  contain LWV's.

**Definition 1.41.** Let  $V$  be an  $\mathfrak{h}$ -diagonalizable  $\mathcal{L}$ -module with weights  $P(V)$ . Define

$$P(V)_+ = \{\mu \in P(V) \mid wt(\mu) \geq 0\}, \quad \text{and} \quad P(V)_- = \{\mu \in P(V) \mid wt(\mu) < 0\},$$

so  $P(V) = P(V)_- \cup P(V)_+$ . Then we define

$$V_{\pm} = \bigoplus_{\mu \in P(V)_{\pm}} V_{\mu},$$

so  $V = V_- \oplus V_+$ .

Lastly, we introduce *non-standard modules*, which will be explored in Chapters 3 – 5. Our choice of properties in part (ii) below was motivated by the fact that highest and lowest weights satisfy properties (2) – (4), but not property (1).

**Definition 1.42.** Let  $V$  be an irreducible, integrable,  $\mathfrak{h}$ -diagonalizable  $\mathcal{L}$ -module. So for any  $0 \neq v \in V$ , we have  $V = \mathcal{U}(\mathcal{L}) \cdot v$ .

- i) We say  $V$  is **standard** if it is a highest- or lowest-weight module. In other words, there exists a unique  $\lambda \in P(V)$  that is either a highest weight (so  $\lambda \in P^+$ ) or a lowest weight (so  $\lambda \in P^-$ ). In either case, we write  $V = V^{\lambda}$ .
- ii) We say  $V$  is **non-standard** if it is not a standard module, so there exists a weight  $\lambda \in P(V)$  such that 1)  $\lambda \notin P^{\pm}$ , 2)  $\|\lambda\|$  is maximal in  $\{\|\mu\| \mid \mu \in P(V)\}$ , 3)  $|wt(\lambda)|$  is minimal among those, and 4)  $V = \mathcal{U}(\mathcal{L}) \cdot v_{\lambda}$  for some  $v_{\lambda} \in V_{\lambda}$ . We may write  $V = V^{\lambda}$  for any  $\lambda$  that satisfies these properties.

Although the label  $\lambda$  for non-standard  $V^{\lambda}$  is not uniquely determined, there are only finitely many weights that satisfy the properties of the definition. Note that  $\mathcal{L}$  itself is a non-standard module  $V^{\alpha_i}$  for any  $\alpha_i \in \Pi$ .

Let  $V^{\lambda}$  be either a standard or non-standard  $\mathcal{L}$ -module. If we identify the set of fundamental weights with a basis for  $\mathbb{R}^{\ell}$  we obtain a visualization of  $P(V^{\lambda})$  called the *weight diagram* of  $V^{\lambda}$ , where each dot corresponds to a weight  $\mu \in P(V^{\lambda})$ , and is labeled with its inner multiplicity. A weight diagram without multiplicities is called an unlabeled weight diagram.

Since  $P(V^{\lambda})$  is a saturated set of weights, we have  $P(V^{\lambda}) = R_1 \cup R_2$  where

$$R_1 = \bigcup_{w \in W} \{S_{\alpha_i}(w\lambda) \mid 1 \leq i \leq \ell\} \quad \text{and} \quad R_2 = \bigcup_{\mu \in R_1} S_{\alpha_i}(\mu)$$

(cf. (1.3) above). This gives a recursive method for determining the weight diagram of  $V^{\lambda}$ . Examples of weight diagrams are shown in Figures 1.3, 4.1b, 4.2b, 4.3b, and 5.2.



### 1.3 The hyperbolic Kac-Moody Lie algebra $\mathcal{F}$

Consider the  $3 \times 3$  the generalized Cartan matrix

$$A = (a_{ij}) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

As outlined in section 1.2,  $A$  has realization  $(\mathfrak{h}, \Pi = \{\alpha_i\}, \Pi^\vee = \{h_i\})_{i=1,2,3}$  (where  $h_i = \alpha_i^\vee$  since  $A$  is symmetric), and to  $A$  we associate a KM algebra  $\mathcal{F} = \mathcal{F}(A)$  over  $\mathbb{C}$ , generated by the elements  $\{h_i, e_i, f_i \mid i = 1, 2, 3\}$  subject to the relations presented in Definition 1.13. We also have:

- the abelian Cartan subalgebra  $\mathfrak{h}$  with basis  $\Pi^\vee$
- root lattice  $Q = Q_{\mathcal{F}} \subset \mathfrak{h}^*$
- root system  $\Phi \subset Q \subset \mathfrak{h}^*$  with basis of simple roots  $\Pi$ .
- $\alpha_i(h_j) = a_{ij}$
- root spaces  $\mathcal{F}_\alpha = \{x \in \mathcal{F} \mid [h, x] = \alpha(h)x, h \in \mathfrak{h}\}$ .
- $\mathcal{F}_{\alpha_i} = \mathbb{C}e_i$  and  $\mathcal{F}_{-\alpha_i} = \mathbb{C}f_i$
- Cartan involution  $\nu = \nu_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F}$  given by  $\nu(e_i) = -f_i$  and  $\nu(h_i) = -h_i$  for  $i = 1, 2, 3$ .
- root space and triangular decompositions,  $\mathcal{F} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathcal{F}_\alpha = \mathfrak{h} \oplus \mathcal{F}^+ \oplus \mathcal{F}^-$
- Weyl group  $W_{\mathcal{F}} = W(A) = \langle w_i \mid 1 \leq i \leq 3 \rangle$
- set of integral weights  $P = \{\omega \in \mathfrak{h}^* \mid \langle \alpha_j, \omega \rangle \in \mathbb{Z}, 1 \leq j \leq 3\}$ ,
- fundamental weights  $\omega_1, \omega_2, \omega_3 \in P$ ,
- dominant integral weights  $P^+ = \{\omega \in P \mid \langle \alpha_j, \omega \rangle \geq 0, 1 \leq j \leq 3\}$

Define the real vector space  $\mathfrak{h}_{\mathbb{R}} = \mathbb{R}h_1 \oplus \mathbb{R}h_2 \oplus \mathbb{R}h_3$  and its dual  $\mathfrak{h}_{\mathbb{R}}^* = \{\beta = x\alpha_1 + y\alpha_2 + z\alpha_3 \in \mathfrak{h}^* \mid x, y, z \in \mathbb{R}\}$  with indefinite non-degenerate quadratic form of signature  $(2,1)$  determined by the GCM  $A$ ,

$$\|\beta\|^2 = [\beta]^t A [\beta] = 2(x^2 - 2xy + y^2 - yz + z^2)$$

where  $[\beta]$  is the coordinate vector of  $\beta$  with respect to  $\Pi$ , so  $\mathfrak{h}_{\mathbb{R}}^* \simeq \mathbb{R}^{(2,1)}$ . For  $c \in \mathbb{R}$ ,

$$S_c = \{\beta \in \mathfrak{h}_{\mathbb{R}}^* \mid \|\beta\|^2 = c\} \tag{1.5}$$

is a quadric surface, either a hyperboloid or a cone (see Figure 1.1 for examples).

If we set  $a = z - y$ ,  $b = x - y$ ,  $c = -z$ , we obtain

$$\|\beta\|^2 = 2(b^2 - ac) = -2 \det \begin{bmatrix} a & b \\ b & c \end{bmatrix}. \quad (1.6)$$

We are therefore motivated to represent any  $\beta \in \mathfrak{h}_{\mathbb{R}}^*$  as a  $2 \times 2$  real symmetric matrix:

$$\beta = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} z-y & x-y \\ x-y & -z \end{bmatrix} = x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + y \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

so the simple roots are:

$$\alpha_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad \alpha_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (1.7)$$

By polarization of the above quadratic form, we obtain the following non-degenerate linear form (in the second entry) on  $\mathfrak{h}_{\mathbb{R}}^*$  represented by  $A$  with respect to the basis  $\Pi$ :

$$\langle \alpha, \alpha' \rangle = [\alpha]^t A [\alpha'] = \frac{1}{2} (\|\alpha + \alpha'\|^2 - \|\alpha\|^2 - \|\alpha'\|^2).$$

For  $\alpha = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ ,  $\alpha' = \begin{bmatrix} a' & b' \\ b' & c' \end{bmatrix} \in \mathfrak{h}_{\mathbb{R}}^*$ , we have

$$\begin{aligned} \langle \alpha, \alpha' \rangle &= \frac{1}{2} (2[(b+b')^2 - (a+c')(c+c')] - 2[b^2 - ac] - 2[b'^2 - a'c']) \\ &= 2bb' - ac' - a'c. \end{aligned} \quad (1.8)$$

Note that for  $i, j = 1, 2, 3$ , this formula gives  $\langle \alpha_i, \alpha_j \rangle = a_{ij} = \alpha_j(h_i)$ .

The Weyl group  $W$  has a presentation as a Coxeter group

$$W = \langle w_1, w_2, w_3 \mid w_i^2 = 1, (w_1 w_3)^2 = (w_2 w_3)^3 = 1 \rangle,$$

which is actually a hyperbolic triangle group and matrix group,

$$W \simeq T(2, 3, \infty) \simeq PGL_2(\mathbb{Z}) = \{M \in \mathbb{Z}_2^2 \mid \det M = \pm 1\} / \{\pm I\}.$$

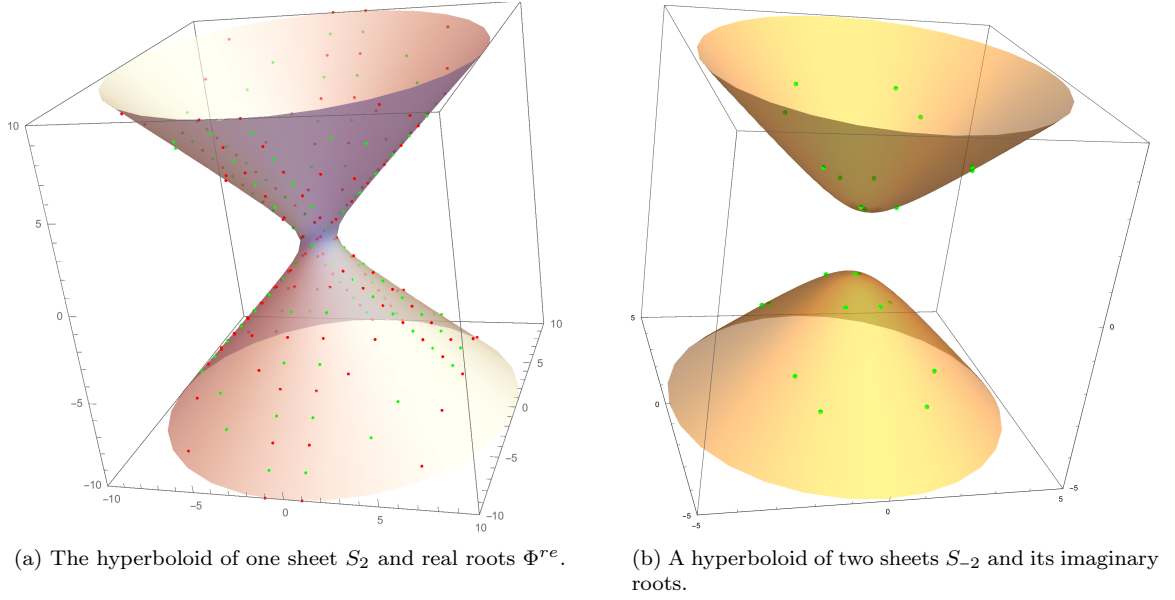
The latter isomorphism will prove especially useful since it will allow us to realize elements of  $W$  by  $2 \times 2$  matrices.

We have the partition of  $\Phi = \Phi^{re} \cup \Phi^{im}$ , where

$$\Phi^{re} = \{\alpha \in Q \mid \|\alpha\|^2 = 2\} \quad \text{and} \quad \Phi^{im} = \{\alpha \in Q \mid \|\alpha\|^2 \leq 0\}.$$

$\Phi^{re}$  is contained in the hyperboloid of one sheet,  $S_2$ , shown in Figure 1.1a. (Note that the roots on this hyperboloid are partitioned into two colors, corresponding to the partition of  $\Phi^{re}$  into two disjoint  $W$ -orbits of  $\Pi$  [CCFP].) Also,

$$\Phi^{im} \subset \bigcup_{n=0}^{\infty} S_{-2n},$$


 Figure 1.1: Quadric surfaces of constant squared-length in  $\mathfrak{h}_{\mathbb{R}}^*$ .

where  $S_0$  is the *null-cone* and  $S_{-2n}$  for  $n \geq 1$  are hyperboloids of two sheets strictly inside the *light-cone*, defined to be

$$LC = \{\alpha \in \mathfrak{h}_{\mathbb{R}}^* \mid (\alpha, \alpha) \leq 0\}.$$

If  $\|\alpha\|^2 = 0$  (so  $\alpha$  lies on the null-cone) then  $Mult(\alpha) = 1$ . A portion of  $S_{-2}$  is shown in Figure 1.1b. The light-cone is partitioned into the *backward* and *forward* light-cones,

$$LC = LC_- \cup LC_+,$$

such that  $\Phi_{\pm}^{im} \subset LC_{\pm}$ . For  $\alpha = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \Phi^{im}$  with  $\|\alpha\|^2 = 2(b^2 - ac) < 0$ , then if  $c > 0$ , then  $\alpha \in \Phi_-^{im} \subset LC_-$ , and if  $c < 0$ , then  $\alpha \in \Phi_+^{im} \subset LC_+$ .

As Figure 1.1b indicates, for each  $n > 0$ , there is a “positive” sheet of  $S_{-n}$  that lies in  $LC_+$  and a “negative” sheet that lies in  $LC_-$ . Since each sheet is  $W$ -invariant, we have that  $W(LC_{\pm}) = LC_{\pm}$ , and

$$W(LC_+) \cap W(LC_-) = \{0\}. \quad (1.9)$$

We can also represent the real and imaginary roots as

$$\Phi^{re} = \{\alpha \in \Phi \mid \det(\alpha) = -1\} \quad \text{and} \quad \Phi^{im} = \{\alpha \in \Phi \mid \det(\alpha) \geq 0\}.$$

The formulas for the simple reflections determine their actions on  $\alpha = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ :

$$w_1\alpha = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}, \quad w_2\alpha = \begin{bmatrix} a - 2b + c & c - b \\ c - b & c \end{bmatrix}, \quad w_3\alpha = \begin{bmatrix} c & b \\ b & a \end{bmatrix}.$$

Defining the three matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we see that for  $i = 1, 2, 3$ ,  $w_i \alpha = M_i \alpha M_i^t$ . If  $w = w_{i_1} \cdots w_{i_s} \in W$  then there is a corresponding  $M = M_{i_1} \cdots M_{i_s} \in PGL_2(\mathbb{Z})$  such that  $w(\alpha) = M \alpha M^t$ .

Using

$$A^{-1} = \begin{pmatrix} -\frac{3}{2} & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & 0 \end{pmatrix},$$

we may write the fundamental weights in terms of the simple roots as follows:

$$\omega_1 = -\frac{3}{2}\alpha_1 - 2\alpha_2 - \alpha_3 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}, \quad \omega_2 = -2\alpha_1 - 2\alpha_2 - \alpha_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\text{and } \omega_3 = -\alpha_1 - \alpha_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus for  $\alpha \in \Phi$ , we have

$$\alpha \in \Phi_{\pm} \text{ if and only if } \langle \alpha, \pm \omega_3 \rangle = \mp c > 0. \quad (1.10)$$

We also observe that if  $\omega = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in P^+$  then  $a, 2b, c \in \mathbb{Z}$  and by definition 1.21,

$$\langle \alpha_1, \omega \rangle = 2b \geq 0, \quad \langle \alpha_2, \omega \rangle = -2b + c \geq 0, \quad \langle \alpha_3, \omega \rangle = a - c \geq 0,$$

giving us  $P^+$  as the intersection of  $P$  with the three real half-spaces defined by the  $\alpha_i^{\perp}$ ,

$$P^+ = \left\{ \omega = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mid a, 2b, c \in \mathbb{Z}, a \geq c \geq 2b \geq 0 \right\}.$$

Note that  $P^+ \subset LC_-$  since for  $\omega \in P^+$ ,

$$\det(\omega) = ac - b^2 \geq (2b)(2b) - b^2 = 3b^2 \geq 0.$$

The (positive) Tits cone  $T_+$  is defined to be the strict interior of the forward light-cone  $LC_+$ , namely,

$$T_+ = \{ \alpha \in LC_+ \mid (\alpha, \alpha) < 0 \}.$$

Define  $P^- = -P^+$ . Then a result of Kac ([K2]) shows that  $P^-$  is a fundamental domain for the action of the Weyl group on  $P \cap LC_+$ , the weight lattice points which lie within the forward light-cone  $LC_+$  (including points on the null cone).

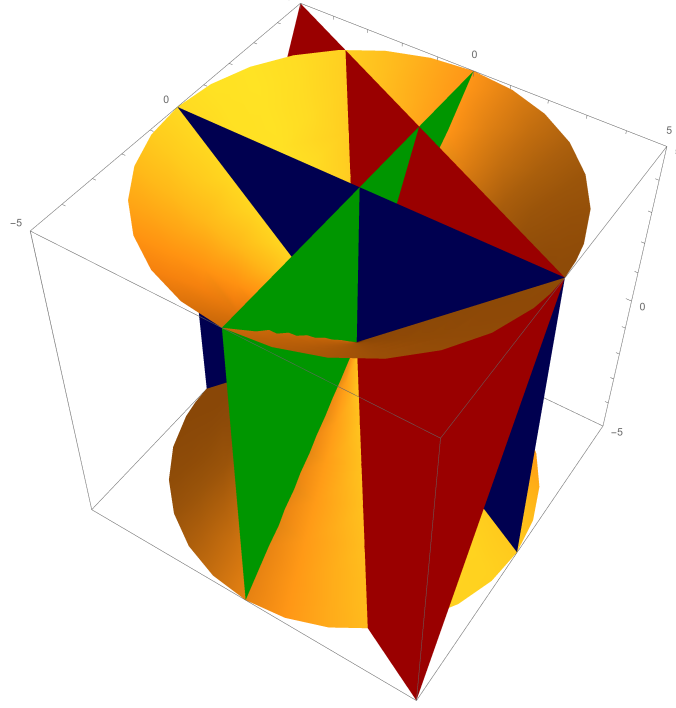


Figure 1.2:  $P^-$  is bounded by the reflecting planes  $\alpha_1^\perp, \alpha_2^\perp, \alpha_3^\perp$ . The null-cone is shown in yellow.

#### 1.4 Feingold-Frenkel decomposition of $\mathcal{F}$ with respect to $A_1^{(1)}$

In [FF], Feingold and Frenkel investigated the decomposition of  $\mathcal{F}$  with respect to a rank 2 affine subalgebra. Consider the generalized Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

and consider the subalgebra of  $\mathcal{F}$  generated by  $\{e_i, f_i, h_i \mid i = 1, 2\}$ . The bilinear form on  $\mathcal{F}$  restricted to this subspace is degenerate, however, so the corresponding simple roots  $\alpha_1, \alpha_2$  are no longer linearly independent. This subalgebra is isomorphic to the central extension of the loop algebra,

$$\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where  $\mathfrak{sl}_2(\mathbb{C})$  is the simple finite-dimensional Lie algebra with basis  $\{e, f, h\}$ . To endow this algebra with a symmetric invariant nondegenerate bilinear form, the usual approach is to extend the Cartan subalgebra by adding the derivation  $d = -t \frac{d}{dt}$ . The resulting *affinization* of  $\mathfrak{sl}_2(\mathbb{C})$ ,

$$\widehat{\mathfrak{sl}_2(\mathbb{C})} = \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

is an affine KM algebra. Feingold and Frenkel then give an embedding of  $\widehat{\mathfrak{sl}_2(\mathbb{C})}$  in  $\mathcal{F}$  defined by

$$h \otimes 1 \mapsto h_1, \quad c \mapsto h_1 + h_2, \quad d \mapsto h_1 + h_2 + h_3,$$

$$e \otimes 1 \mapsto e_1, \quad f \otimes 1 \mapsto f_1, \quad f \otimes t \mapsto e_2, \quad e \otimes t^{-1} \mapsto f_2.$$

We refer to this subalgebra as  $\mathcal{A}ff$ . Note the Cartans of  $\mathcal{A}ff$  and  $\mathcal{F}$  are the same.

The simple roots  $\Pi_{\mathcal{A}ff} = \{\alpha_1, \alpha_2\}$  immediately give us

- the affine plane  $\mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2 \subset \mathfrak{h}_{\mathbb{R}}^*$ ,
- the affine root sublattice  $Q_{\mathcal{A}ff} = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ ,
- the affine Weyl group  $W_{\mathcal{A}ff} = \langle w_1, w_2 \rangle < W$ ,
- the affine null root  $\delta = \alpha_1 + \alpha_2$ ,
- the affine root subsystem  $\Phi_{\mathcal{A}ff} = \{n\delta \mid 0 \neq n \in \mathbb{Z}\} \cup \{m\delta \pm \alpha_1 \mid m \in \mathbb{Z}\} \subset \Phi$ ,
- the affine real roots,  $\Phi_{\mathcal{A}ff}^r = W_{\mathcal{A}ff}\Pi_{\mathcal{A}ff} = \{m\delta \pm \alpha_1 \mid m \in \mathbb{Z}\} = \{\alpha \in Q_{\mathcal{A}ff} \mid \|\alpha\|^2 = 2\}$ ,
- the affine imaginary roots  $\Phi_{\mathcal{A}ff}^i = \{n\delta \mid 0 \neq n \in \mathbb{Z}\} = \{\alpha \in Q_{\mathcal{A}ff} \mid \|\alpha\|^2 = 0\}$ ,
- the affine fundamental weights  $\{\omega_1, \omega_2\}$ ,
- the affine weight sublattice  $P_{\mathcal{A}ff} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset P$ , and
- the affine fundamental domain  $P_{\mathcal{A}ff}^\pm = P^\pm \cap (\mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2)$ .

Feingold and Frenkel showed that  $\mathcal{F}$  can be decomposed into an infinite sum of  $\mathcal{A}ff$ -modules, graded by level. The levels of  $\mathcal{F}$  with respect to  $\mathcal{A}ff$  were determined as follows. The affine plane is level 0, and it contains roots of the form

$$n_1\alpha_1 + n_2\alpha_2 = \begin{bmatrix} -n_2 & n_1 - n_2 \\ n_1 - n_2 & 0 \end{bmatrix}.$$

The non-zero levels were determined by adding multiples of the root  $\gamma = \alpha_1 + \alpha_2 + \alpha_3 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$  to the affine plane, so the “level  $m$  affine plane” is  $\mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2 + m\gamma$ , and the root system is partitioned as  $\Phi = \bigcup_{m \in \mathbb{Z}} \Delta_m$ , where

$$\Delta_m = \Delta_{\mathcal{A}ff, m} = \Phi \cap (\mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2 + m\gamma) = \left\{ \beta \in \Phi \mid \beta = \begin{bmatrix} a & b \\ b & -m \end{bmatrix}, a, b, m \in \mathbb{Z} \right\}$$

is the set of all affine level  $m$  roots. The inner product gives a useful method for determining the level of any root in  $\Phi$ . Let  $\delta = (\alpha_1 + \alpha_2) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ . Then

$$\langle -\delta, \alpha \rangle = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a & b \\ b & -m \end{bmatrix} \right\rangle = m.$$

We refer to the root spaces of level  $m$  as  $\mathcal{Aff}(m)$ , thus giving the  $\mathbb{Z}$ -grading mentioned above as

$$\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{Aff}(m).$$

For each  $m \in \mathbb{Z}$ , the action of  $\mathcal{Aff}$  preserves  $\mathcal{Aff}(m)$ , making  $\mathcal{Aff}(m)$  an  $\mathcal{Aff}$ -module. By Kac [K2], each slice  $\mathcal{Aff}(m)$  therefore has a decomposition into a direct sum of irreducible  $\mathcal{Aff}$ -modules. Additionally, each irreducible module  $V$  has a weight space decomposition

$$V = \bigoplus_{\mu \in \Delta_m} V_\mu, \quad \text{where} \quad V_\mu = V \cap \mathcal{F}_\mu.$$

The adjoint representation is the only non-standard  $\mathcal{Aff}$ -module occurring in this decomposition of  $\mathcal{F}$ . (We will see more examples of non-standard modules in the decomposition with respect to  $\mathcal{Fib}$  in later chapters.) Besides the adjoint representation, each irreducible integrable module  $V$  of the decomposition of  $\mathcal{F}$  with respect to  $\mathcal{Aff}$  is standard.

Kac showed that if  $\mathfrak{g}$  is an affine KM algebra of rank  $l + 1$  then the multiplicity of every imaginary root of  $\mathfrak{g}$  is  $l$  [K2]. Therefore the dimension of every root space in  $\mathcal{Aff}(0)$  is 1. Dimensions of the weight spaces in  $\mathcal{Aff}(1)$ , whose dominant integral weights are of the

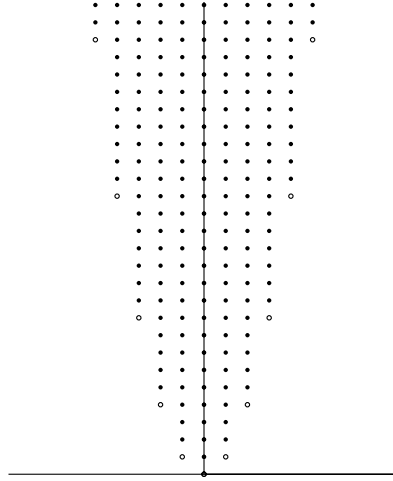


Figure 1.3:  $\mathcal{Aff}(1)$ . Weights of the fundamental domain lie on the center line.

form  $\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$  where  $n \geq -1$  (see Figure 1.3), had already been discovered by Feingold and Lepowsky [FL] to be

$$Mult \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} = p(n+1),$$

where  $p(n)$  is the classical partition function, with generating function  $\sum_{m \geq 0} p(m)q^m = \prod_{k \geq 1} (1 - q^k)^{-1}$ . The particular geometry of this module was the key to the proof, and the fact that all dominant integral weights occur on the central line simplified the Racah-Speiser recursion

to Euler's recursion for the classical partition function. In addition,  $\mathcal{Aff}(1)$  was found to be a single irreducible  $\mathcal{Aff}$ -module, thus the irreducible decompositions for levels 0 and  $\pm 1$  were completely determined.

Of course, as the level increases, so does the size of the dominant integral region and hence the complexity of the problem. In  $\mathcal{Aff}(2)$ , [FF] found the fundamental domain to consist of two lines of roots, of the form  $\begin{pmatrix} n & 0 \\ 0 & 2 \end{pmatrix}$  and  $\begin{pmatrix} n & 1 \\ 1 & 2 \end{pmatrix}$  for  $n \geq 0$ . Here, the vertex operator construction of  $\mathcal{Aff}(1)$  from [LW] was used to decompose the tensor product  $\mathcal{Aff}(1) \otimes \mathcal{Aff}(1)$  into symmetric and antisymmetric tensors. Further analysis of the antisymmetric tensors that correspond to commutators in  $[\mathcal{Aff}(1), \mathcal{Aff}(1)]$  resulted in a complete determination of  $\mathcal{Aff}(2)$  root multiplicities,

$$\text{Mult} \begin{pmatrix} n & 0 \\ 0 & 2 \end{pmatrix} = p'(2n+1) \quad \text{and} \quad \text{Mult} \begin{pmatrix} n & 1 \\ 1 & 2 \end{pmatrix} = p'(2n)$$

where  $p'(m)$  is a 'modified partition function' whose generating function

$$\sum_{m \geq 0} p'(m) q^m = \left( \prod_{k \geq 1} (1 - q^k)^{-1} \right) (1 - q^{20} + q^{22} - \dots)$$

agrees with the classical function for the first 20 terms.

This method of tensor decomposition of higher levels with respect to  $\mathcal{Aff}$  was then used by Kang in several papers of the 1990s (e.g. [Ka1], [Ka2]), eventually giving results up to level 5. The question for higher levels is still open, as is finding a closed form formula for the multiplicity of an arbitrary root of  $\mathcal{F}$ .



## Chapter 2 The Fibonacci subalgebra $\mathcal{Fib}$ of $\mathcal{F}$

As an alternative to finding a decomposition of  $\mathcal{F}$  with respect to  $\mathcal{Aff}$ , it is possible to decompose  $\mathcal{F}$  with respect to other rank 2 subalgebras inside  $\mathcal{F}$ . We wish to explore how  $\mathcal{F}$  decomposes with respect to a particular rank 2 subalgebra of hyperbolic type. We will show that one of the crucial differences is the emergence of several non-standard modules, which are irreducible and integrable like the adjoint representation, but are not generated by either a highest- or lowest-weight vector. The affine decomposition of  $\mathcal{F}$  does not contain non-standard modules, except for the adjoint. We will discuss non-standard modules briefly in this section, and explore them in greater depth in Chapter 5.

Feingold and Nicolai [FN] showed that all of the rank 2 hyperbolic KM Lie algebras, whose GCM are given by

$$B(n) = \begin{pmatrix} 2 & -n \\ -n & 2 \end{pmatrix}$$

for  $n \geq 3$  and which we denote by  $\mathcal{H}(n)$ , are contained in  $\mathcal{F}$ . In other words, there exist roots  $\beta_1, \beta_2 \in \Phi^{re}$  such that  $B(n) = (\langle \beta_i, \beta_j \rangle)$ . Also, there exist Serre generators

$$E_i = E_i(n) \in \mathcal{F}_{\beta_i}, \quad F_i = F_i(n) \in \mathcal{F}_{-\beta_i}, \quad \text{and} \quad H_i = H_i(n) = [E_i, F_i] \in \mathfrak{h}$$

of an algebra isomorphic to  $\mathcal{H}(n)$ , whose Cartan subalgebra is  $\mathfrak{h}(n) = \mathbb{C}H_1 \oplus \mathbb{C}H_2$ . We use the notation  $\mathfrak{h}_{\mathbb{R}}(n) = \mathbb{R}H_1 \oplus \mathbb{R}H_2$ . (Recall from the previous section that  $\mathfrak{h}_{\mathcal{Aff}} = \mathfrak{h}$ , while  $\mathfrak{h}(n) \subsetneq \mathfrak{h}$ .) Furthermore, the dual  $\mathfrak{h}(n)_{\mathbb{R}}^*$  has inner product  $(\cdot, \cdot)_n$  determined by  $B(n)$  which agrees with the inner product on  $\mathfrak{h}_{\mathbb{R}}^*$  determined by  $A$ .

The simplest case is when  $n = 3$ , giving the Cartan matrix

$$B(3) = (b_{ij}) = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$$

which corresponds to  $\mathcal{H}(3)$ , the simplest rank 2 KM algebra of hyperbolic type. Feingold [F1] discovered that the Fibonacci numbers occur in an interesting way in the Weyl-Kac denominator formula for this algebra. We therefore refer to  $\mathcal{H}(3)$  as the *Fibonacci hyperbolic*.

In order to decompose  $\mathcal{F}$  with respect to a subalgebra isomorphic to  $\mathcal{H}(3)$ , we must first find a realization of  $\mathcal{H}(3)$  inside  $\mathcal{F}$ . The resulting subalgebra will be denoted by  $\mathcal{Fib}$ . First,

we find two positive real root vectors whose corresponding roots  $\beta_1, \beta_2 \in \Phi^{re}$  are simple roots of  $\mathcal{H}(3)$ . Following [FN] we could choose  $\beta_1$  and  $\beta_2$  to be

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \alpha_3 \quad \text{and} \quad \begin{bmatrix} -3 & \pm 1 \\ \pm 1 & 0 \end{bmatrix},$$

respectively, so  $\beta_2 = 2\alpha_1 + 3\alpha_2$  or  $2\alpha_1 + 4\alpha_2$ . However, there is a choice whereby  $\beta_1$  and  $\beta_2$  have lower combined height, which will be helpful for multibracket and vertex algebra calculations in later sections. Similar to [FN] we choose one root to be in  $\Pi$ . We cannot choose  $\alpha_1$  since for any other  $\alpha \in \Phi^{re}$  we get

$$\langle \alpha_1, \alpha \rangle = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right\rangle = 2b \neq -3,$$

since  $b \in \mathbb{Z}$ . Choosing  $\alpha_2$  gives the condition

$$\langle \alpha_2, \alpha \rangle = \left\langle \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right\rangle = -2b + c = -3,$$

and  $\alpha \in \Phi^{re}$  gives the additional condition  $ac - b^2 = -1$ . Setting  $a = 0$  and  $b = 1$  yields  $c = -1$ , hence  $\alpha = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$  is a valid candidate for the second simple root.

In the following proposition, we use the multibracket notation from Definition 1.38.

**Proposition 2.1.** *Let  $\beta_1 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = 2\alpha_1 + \alpha_2 + \alpha_3$ ,  $\beta_2 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} = \alpha_2 \in \Phi$ . Then  $\beta_1, \beta_2$  are simple roots of a subalgebra of  $\mathcal{F}$ , denoted by  $\mathcal{Fib}$ , which is isomorphic to  $\mathcal{H}(3)$ . The elements*

$$E_1 = \frac{1}{2}e_{1123} \in \mathcal{F}_{\beta_1}, \quad F_1 = -\frac{1}{2}f_{1123} \in \mathcal{F}_{-\beta_1}, \quad E_2 = e_2 \in \mathcal{F}_{\beta_2}, \quad F_2 = f_2 \in \mathcal{F}_{-\beta_2},$$

$$H_1 = 2h_1 + h_2 + h_3, \quad H_2 = h_2 \in \mathfrak{h}$$

are Serre generators of  $\mathcal{Fib}$ . We denote the  $\mathbb{C}$ -span of  $H_1, H_2$  by  $\mathfrak{h}_{\mathcal{Fib}}$ , the  $\mathbb{R}$ -span of  $H_1, H_2$  by  $(\mathfrak{h}_{\mathcal{Fib}})_{\mathbb{R}}$ , and the  $\mathcal{Fib}$  subroot system of  $\Phi$  by  $\Delta = \Delta_{\mathcal{Fib}}$ . Also,  $\nu$  restricts to the Cartan involution of  $\mathcal{Fib}$ , which we also denote by  $\nu$ , in other words,  $\nu(E_i) = -F_i$  and  $\nu(H_i) = -H_i$  for  $i = 1, 2$ .

*Proof.* We show these generators satisfy the Serre relations of  $\mathcal{H}(3)$ .

1) Clearly  $H_1$  and  $H_2$  commute.

2) We have  $[E_1, F_2] = -\frac{1}{2}[f_2, e_{1123}] = 0$ ,  $[E_2, F_1] = \frac{1}{2}[e_2, f_{1123}] = 0$ , and  $[E_2, F_2] = [e_2, f_2] = h_2 = H_2$ . The next calculation shows that  $[E_1, F_1] = H_1$ , and makes use of the multibracket identities that follow from Theorem B.1, listed in Appendix B.

$$[E_1, F_1] = -\frac{1}{4}[e_{1123}, f_{1123}] = -\frac{1}{4}([e_1, f_{1123}], e_{123}] + [e_1, [e_{123}, f_{1123}]])$$

$$\begin{aligned}
 &= -\frac{1}{4} \left( [2f_{123}, e_{123}] + [e_1, [[e_{123}, f_1], f_{123}] + [f_1, [e_{123}, f_{123}]]] \right) \\
 &= \frac{1}{4} \left( 2[e_1, [e_{23}, f_{123}]] + 2[[e_1, f_{123}], e_{23}] + 2[e_1, [e_{23}, f_{123}]] \right. \\
 &\quad \left. - [e_1, [f_1, [e_1, [e_{23}, f_{123}]]] + [[e_1, f_{123}], e_{23}]] \right) \\
 &= \frac{1}{4} \left( 4[e_1, [e_2, [e_3, f_{123}]]] + [[e_2, f_{123}], e_3] - 4[e_{23}, f_{23}] \right. \\
 &\quad \left. - [e_1, [f_1, [e_1, [e_2, [e_3, f_{123}]]] + [[e_2, f_{123}], e_3]]] - 2[e_1, [f_1, [f_{23}, e_{23}]]] \right) \\
 &= \frac{1}{4} \left( -4[e_1, [e_2, f_{12}]] + 4[f_2, [f_3, e_{23}]] + 4[[f_2, e_{23}], f_3] \right. \\
 &\quad \left. + [e_1, [f_1, [e_1, [e_2, f_{12}]]]] - 2[e_1, [f_1, [f_2, -e_2] + [e_3, f_3]]] \right) \\
 &= \frac{1}{4} \left( -4[e_1, -2f_1] + 4[f_2, -e_2] + 4[e_3, f_3] + [e_1, [f_1, [e_1, -2f_1]]] - 2[e_1, [f_1, h_2 + h_3]] \right) \\
 &= \frac{1}{4} \left( 8h_1 + 4h_2 + 4h_3 + [e_1, [f_1, -2h_1]] - 2[e_1, -2f_1 + 0] \right) \\
 &= \frac{1}{4} \left( 8h_1 + 4h_2 + 4h_3 - [e_1, -4f_1 + 4f_1] \right) = 2h_1 + h_2 + h_3,
 \end{aligned}$$

which is exactly  $H_1$ . We also have  $[E_2, F_2] = [e_2, f_2] = h_2 = H_2$ .

3) The following calculations show that  $[H_i, E_j] = b_{ji}E_j$  for  $i, j = 1, 2$ .

$$[H_1, E_1] = [2h_1 + h_2 + h_3, E_1] = \beta_1(2h_1 + h_2 + h_3)E_1 = (2\alpha_1 + \alpha_2 + \alpha_3)(2h_1 + h_2 + h_3)E_1 = 2E_1,$$

$$[H_1, E_2] = [2h_1 + h_2 + h_3, e_2] = \beta_2(2h_1 + h_2 + h_3)e_2 = \alpha_2(2h_1 + h_2 + h_3)e_2 = -3e_2 = -3E_2,$$

$$[H_2, E_1] = [h_2, E_1] = \beta_1(h_2)E_1 = (2\alpha_1 + \alpha_2 + \alpha_3)(h_2)E_1 = -3E_1, \quad \text{and} \quad [H_2, E_2] = 2E_2.$$

4) It can easily be shown based on (3) above that  $[H_i, F_j] = -b_{ji}F_j$ .

5) We now show  $(ad_{E_j})^{1-b_{ij}}E_j = 0$  for  $1 \leq i \neq j \leq 2$ . First, we have that

$$[E_2, E_1] \in \mathcal{F}_{2\alpha_1+2\alpha_2+\alpha_3}, \quad [E_2, [E_2, E_1]] \in \mathcal{F}_{2\alpha_1+3\alpha_2+\alpha_3}, \quad [E_2, [E_2, [E_2, E_1]]] \in \mathcal{F}_{2\alpha_1+4\alpha_2+\alpha_3}$$

are all root space vectors, since the squared-lengths of their corresponding roots are:

$$\|2\alpha_1 + 2\alpha_2 + \alpha_3\| = 2(4 - 8 + 4 - 2 + 1) = -2, \quad \|2\alpha_1 + 3\alpha_2 + \alpha_3\| = 2(4 - 12 + 9 - 3 + 1) = -2,$$

$$\text{and} \quad \|2\alpha_1 + 4\alpha_2 + \alpha_3\| = 2(4 - 16 + 16 - 4 + 1) = 2.$$

However,  $[E_2, [E_2, [E_2, [E_2, E_1]]]]$  is not a root space vector since  $\|2\alpha_1 + 5\alpha_2 + \alpha_3\| = 2(4 - 20 + 25 - 5 + 1) = 10$ , therefore  $2\alpha_1 + 5\alpha_2 + \alpha_3 \notin \Delta$  and  $(ad_{E_2})^4 E_1 = 0$ .

Similarly,

$$[E_1, E_2] \in \mathcal{F}_{2\alpha_1+2\alpha_2+\alpha_3}, \quad [E_1, [E_1, E_2]] \in \mathcal{F}_{4\alpha_1+3\alpha_2+2\alpha_3}, \quad [E_1, [E_1, [E_1, E_2]]] \in \mathcal{F}_{6\alpha_1+4\alpha_2+3\alpha_3}$$

are root space vectors since

$$\|2\alpha_1 + 2\alpha_2 + \alpha_3\| = 2(4 - 8 + 4 - 2 + 1) = -2, \quad \|4\alpha_1 + 3\alpha_2 + 2\alpha_3\| = 2(16 - 24 + 9 - 6 + 4) = -2,$$

$$\text{and } \|6\alpha_1 + 4\alpha_2 + 3\alpha_3\| = 2(36 - 48 + 16 - 12 + 9) = 2,$$

but  $[E_1, [E_1, [E_1, [E_1, E_2]]]] \in \mathcal{F}_{8\alpha_1+5\alpha_2+4\alpha_3}$  is not since  $\|8\alpha_1 + 5\alpha_2 + 4\alpha_3\| = 2(64 - 80 + 25 - 20 + 16) = 10$ , giving us  $(ad_{E_1})^4 E_2 = 0$ .

6) Multibracket calculations involving  $F_1$  and  $F_2$  similar to (5) above will show that  $(ad_{F_i})^{-a_{ij}+1}(F_j) = 0$ ,  $i \neq j$ .

The rest is obvious.  $\square$

The proposition allows us to identify the dual Cartan subalgebra of  $\mathcal{H}(3)$  with the dual Cartan  $\mathfrak{h}_{\mathcal{Fib}}^*$  of  $\mathcal{Fib}$ . As before, we call  $(\mathfrak{h}_{\mathcal{Fib}}^*)_{\mathbb{R}} = \mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2$  the  $\mathcal{Fib}$ -plane. The roots of  $\mathcal{Fib}$ ,  $\Delta$ , lie in the  $\mathcal{Fib}$  root lattice,  $Q_{\mathcal{Fib}} = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$ .

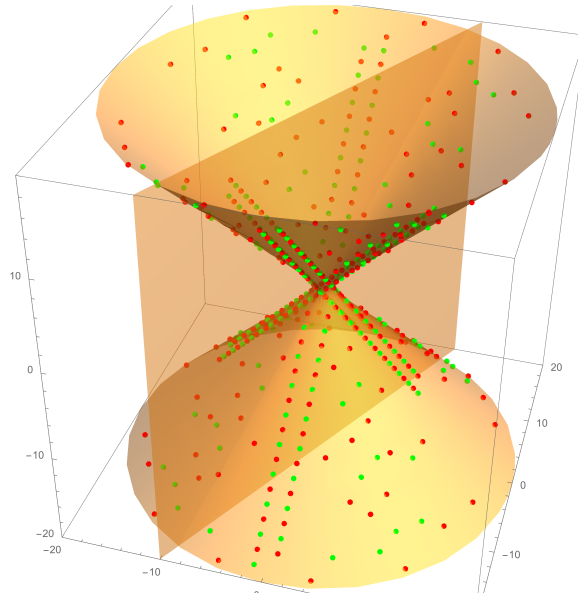


Figure 2.1:  $\mathcal{Fib}$ -plane  $\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2$  ‘slicing’ through  $S_2$ , the real root hyperboloid of  $\mathcal{F}$ . Shown are real roots of  $\mathcal{F}$ . The two colors correspond to the two Weyl orbits of  $\Phi^{re}$ .

**Proposition 2.2.** *We have  $Q_{\mathcal{Fib}} = Q \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2)$ .*

*Proof.* The containment  $\subseteq$  is obvious. For the reverse containment, let  $\alpha \in Q \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2)$ . Then  $\alpha = n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 = c_1\beta_1 + c_2\beta_2$ , where each  $n_i \in \mathbb{Z}$  and each  $c_i \in \mathbb{R}$ . Then

$$n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 = 2c_1\alpha_1 + (c_1 + c_2)\alpha_2 + c_1\alpha_3$$

implies that  $c_1, c_2 \in \mathbb{Z}$ , so  $\alpha \in Q_{\mathcal{Fib}}$ .  $\square$

Since  $(\mathfrak{h}_{\mathcal{Fib}}^*)_{\mathbb{R}}$  inherits the bilinear form from  $\mathfrak{h}_{\mathbb{R}}^*$ , we have

$$\Delta^{re} = \{\beta \in Q_{\mathcal{Fib}} \mid \|\beta\|^2 = 2\} = \Phi^{re} \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2),$$

$$\Delta^{im} = \{\beta \in Q_{\mathcal{Fib}} \mid \|\beta\|^2 \leq 0\} = \Phi^{im} \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2),$$

and

$$\Delta = \Phi \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2).$$

For any root  $\beta \in \Delta$ ,

$$\beta = n_1\beta_1 + n_2\beta_2 = \begin{bmatrix} -n_2 & n_1 - n_2 \\ n_1 - n_2 & -n_1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

for some integers  $n_1, n_2$  both non-negative or non-positive, so  $a = b + c$  describes the  $\mathcal{Fib}$ -plane (for  $a, b, c \in \mathbb{R}$ ).

Define  $\lambda_1, \lambda_2 \in \mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2$  to be fundamental weights for  $\mathcal{Fib}$ , in other words,

$$\langle \beta_i, \lambda_j \rangle = \lambda_j(H_i) = \delta_{ij}. \quad (2.1)$$

Since  $B(3)^{-1} = -\frac{1}{5} \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ , we have

$$\lambda_1 = -\frac{1}{5}(2\beta_1 + 3\beta_2) = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -\frac{1}{5}(3\beta_1 + 2\beta_2) = \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

Denote the weight lattice determined by these fundamental weights by

$$P_{\mathcal{Fib}} = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2.$$

We have the simple reflections  $r_1 = w_1w_2w_3w_2w_1$  and  $r_2 = w_2$ , since

$$\begin{aligned} r_1\beta_1 &= w_1w_2w_3w_2w_1(2\alpha_1 + \alpha_2 + \alpha_3) = w_1w_2w_3w_2(\alpha_2 + \alpha_3) = w_1w_2w_3(\alpha_3) \\ &= w_1w_2(-\alpha_3) = -w_1(\alpha_2 + \alpha_3) = -(2\alpha_1 + \alpha_2 + \alpha_3) = -\beta_1 = \beta_1 - \langle \beta_1, \beta_1 \rangle \beta_1, \\ r_1\beta_2 &= w_1w_2w_3w_2w_1\alpha_2 = w_1w_2w_3w_2(2\alpha_1 + \alpha_2) = w_1w_2w_3(2\alpha_1 + 3\alpha_2) \\ &= w_1w_2(2\alpha_1 + 3\alpha_2 + 3\alpha_3) = w_1(2\alpha_1 + 4\alpha_2 + 3\alpha_3) = (6\alpha_1 + 4\alpha_2 + 3\alpha_3) \\ &= 3\beta_1 + \beta_2 = \beta_2 - \langle \beta_1, \beta_2 \rangle \beta_1, \\ r_2\beta_2 &= w_2\alpha_2 = -\alpha_2 = -\beta_2 = \beta_2 - \langle \beta_2, \beta_2 \rangle \beta_2, \\ r_2\beta_1 &= w_2(2\alpha_1 + \alpha_2 + \alpha_3) = 2\alpha_1 + 4\alpha_2 + \alpha_3 = \beta_1 + 3\beta_2 = \beta_1 - \langle \beta_2, \beta_1 \rangle \beta_2. \end{aligned}$$

Thus the matrices corresponding to  $r_1, r_2$  (as in section 1.3) are

$$M_{r_1} = M_1M_2M_3M_2M_1 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad M_{r_2} = M_2 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The Weyl group  $W_{\mathcal{Fib}} = \langle r_1, r_2 \rangle < W$  permutes the roots  $\Delta$  which is a subset of the  $\mathcal{Fib}$ -plane  $\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2 \simeq \mathbb{R}^{(1,1)} \subset \mathbb{R}^{(2,1)} \simeq \mathfrak{h}_{\mathbb{R}}^*$ . We also have that  $\Delta^{re} = W_{\mathcal{Fib}}\Pi_{\mathcal{Fib}}$  and  $\Delta^{im} = \Delta \setminus \Delta^{re}$ .

We have the sets of real fixed points in  $\mathfrak{h}_{\mathbb{R}}^*$  for  $r_1, r_2$ ,

$$\beta_i^\perp = \{\beta \in \mathfrak{h}_{\mathbb{R}}^* \mid (\beta, \beta_i) = 0\}$$

whose intersection with the  $\mathcal{Fib}$ -plane are the reflecting lines for  $W_{\mathcal{Fib}}$ ,

$$\beta_1^\perp \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2) = \mathbb{R}\lambda_2 \quad \text{and} \quad \beta_2^\perp \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2) = \mathbb{R}\lambda_1.$$

We also have that  $\beta_2^\perp = \alpha_2^\perp$ . The reflecting planes  $\alpha_1^\perp$  and  $\alpha_3^\perp$  intersect the  $\mathcal{Fib}$ -plane in the same line,  $\mathbb{R}(\lambda_1 + \lambda_2) = \alpha_1^\perp \cap \alpha_3^\perp$ .

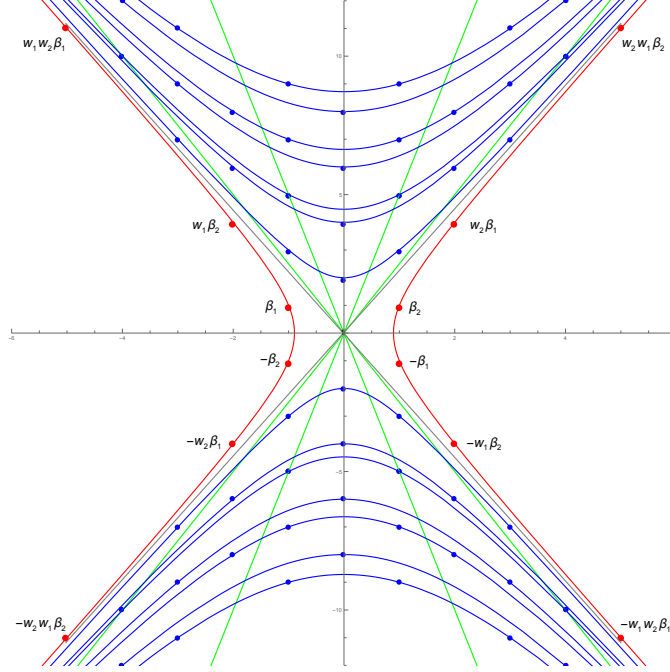


Figure 2.2: The  $\mathcal{Fib}$  root system  $\Delta$ . Real roots lie on the red hyperbola, imaginary roots lie on the blue hyperbolas. The inner green lines show the reflecting planes  $\mathbb{R}\lambda_1, \mathbb{R}\lambda_2$ . The gray lines are the  $\mathcal{Fib}$  null-cone.

The curves of constant squared-length in the  $\mathcal{Fib}$ -plane are

$$S_{\mathcal{Fib},c} = S_c \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2).$$

$S_{\mathcal{Fib},0}$  is the  $\mathcal{Fib}$  null-cone, a pair of lines that intersect at the origin. If  $\mu = c_1\beta_1 + c_2\beta_2 \in S_{\mathcal{Fib},0}$ , then  $\langle \mu, \mu \rangle = 2c_1^2 - 3c_1c_2 + 2c_2^2 = 0$ . Dividing both sides by  $c_1^2$  gives  $(\frac{c_2}{c_1})^2 - 3(\frac{c_2}{c_1}) + 1 = 0$ . The solution  $\frac{c_2}{c_1} = \frac{3 \pm \sqrt{5}}{2}$  shows that the null-cone lines have irrational slope, therefore no roots of  $\mathcal{Fib}$  lie on the null-cone.

For each even  $n \leq 2$ ,  $n \neq 0$ ,  $S_{\mathcal{Fib},n}$  is a disjoint union of two branches of a hyperbola. Each branch of the *real hyperbola*  $S_{\mathcal{Fib},2}$  contains roots of both orbits of  $\Delta^{re} = W_{\mathcal{Fib}}\{\beta_1\} \cup W_{\mathcal{Fib}}\{\beta_2\}$  ([F1]). We have the *Fib light-cone*

$$LC_{\mathcal{Fib}} = LC \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2)$$

which partitions into the *forward* and *backward*  $\mathcal{Fib}$  light-cones,

$$LC_{\mathcal{Fib},\pm} = LC_{\pm} \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2),$$

and the positive  $\mathcal{Fib}$  Tits cone,

$$T_{\mathcal{Fib},+} = T_+ \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2).$$

Figure 2.2 indicates, for each even  $n > 0$ , there is a “positive” branch of  $S_{\mathcal{Fib},-n}$  that lies in  $LC_{\mathcal{Fib},+}$  and a “negative” branch that lies in  $LC_{\mathcal{Fib},-}$ . Since each branch is  $W_{\mathcal{Fib}}$ -invariant, we have that  $W_{\mathcal{Fib}}(LC_{\mathcal{Fib},\pm}) = LC_{\mathcal{Fib},\pm}$ , and

$$W_{\mathcal{Fib}}(LC_{\mathcal{Fib},+}) \cap W_{\mathcal{Fib}}(LC_{\mathcal{Fib},-}) = \{0\}. \quad (2.2)$$

## Chapter 3 *Fib*-modules in $\mathcal{F}$

Our main goal is to study the decomposition of  $\mathcal{F}$  with respect to *Fib* that is analogous to the Feingold-Frenkel decomposition of  $\mathcal{F}$  into *Aff*-modules outlined in Section 1.4. We first introduce the similar notion of *Fibonacci level* by slicing  $\Phi$  into planes parallel to the *Fib*-plane, and show that  $\mathcal{F}$  has a decomposition into *Fib*-modules, graded by Fibonacci level. From this point onwards “level” shall refer to the Fibonacci level. The next two chapters will then explore the decompositions of certain levels in greater detail.

### 3.1 The level $m$ *Fib*-module $\mathcal{F}ib(m)$ , $m \in \mathbb{Z}$

The *level  $m$  Fib-plane* is  $\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2 + m\gamma_{\mathcal{F}ib}$  where  $\gamma_{\mathcal{F}ib} = -\alpha_1 - \alpha_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ . Then for each  $m \in \mathbb{Z}$ , the level  $m$  roots are

$$\Delta_m = \Delta_m = \left\{ \beta \in \Phi \mid \beta = \begin{bmatrix} b+c+m & b \\ b & c \end{bmatrix}, b, c, m \in \mathbb{Z} \right\}.$$

Moreover, given any root  $\alpha = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \Phi$ , we have  $\alpha \in \Delta_{a-b-c}$ , thus we have the disjoint union

$$\Phi = \bigcup_{m \in \mathbb{Z}} \Delta_m.$$

It is clear that if  $\beta \in \Delta_m$  then  $-\beta \in \Delta_{-m}$  so that  $\Delta_{-m} = -\Delta_m$ .

As in Section 1.4, we use the inner product to determine the level of any root in  $\Phi$ . Let  $\delta_{\mathcal{F}ib} = \alpha_3 - \frac{1}{2}\alpha_1 = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{bmatrix} \in P$ . Then for any  $\beta \in \Delta_m$ ,

$$\langle \delta_{\mathcal{F}ib}, \beta \rangle = \left\langle \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{bmatrix}, \begin{bmatrix} b+c+m & b \\ b & c \end{bmatrix} \right\rangle = 2\left(-\frac{1}{2}\right)b - c - (-1)(b+c+m) = m.$$

We define *Fib-level  $m$*  to be

$$\mathcal{F}ib(m) = \bigoplus_{\beta \in \Delta_m} \mathcal{F}_\beta. \quad (3.1)$$

The levels provide a  $\mathbb{Z}$ -grading of  $\mathcal{F}$  according to planes parallel to the *Fib*-plane,

$$\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}ib(m). \quad (3.2)$$



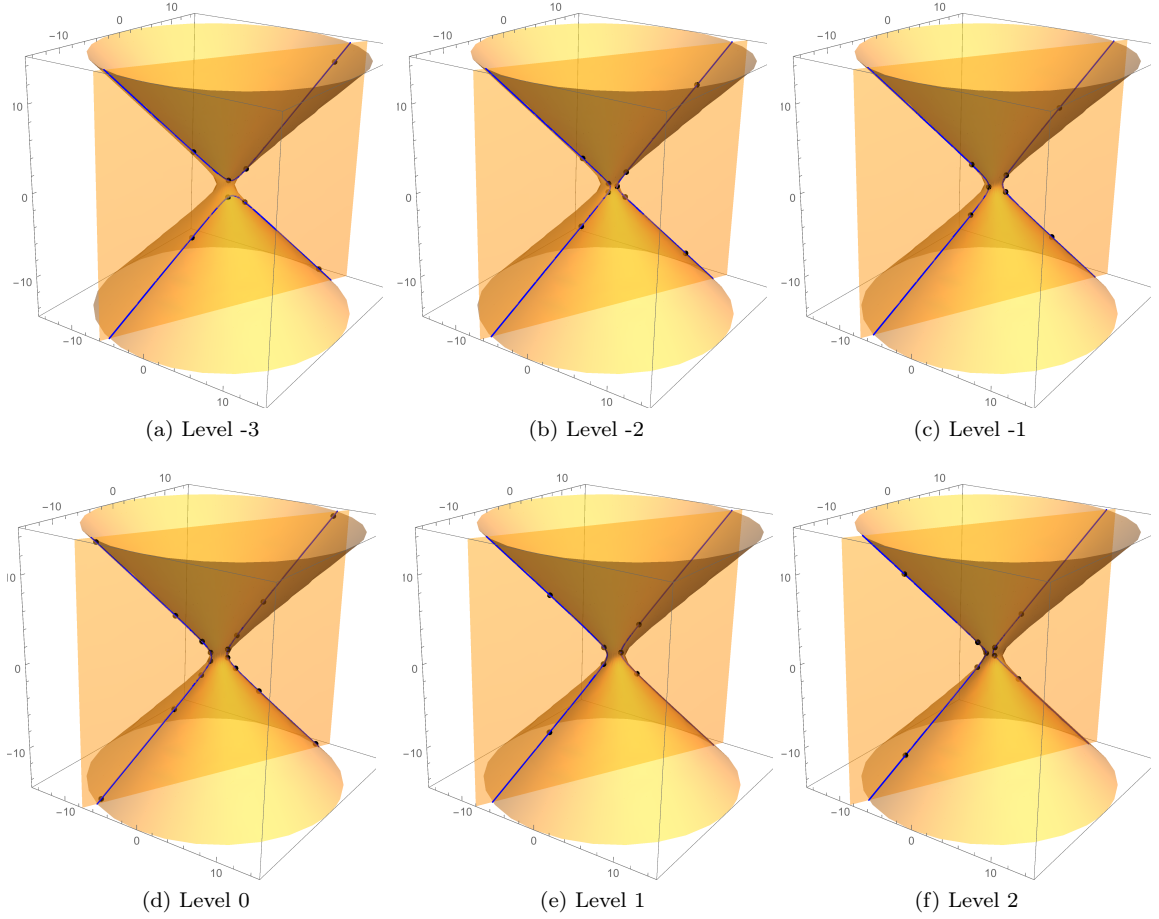


Figure 3.1:  $\mathcal{Fib}$  levels -3 through 2 slicing through the hyperboloid  $S_2$ . Imaginary roots on each level are not shown. Real roots lie on the blue hyperbolas  $S_2 \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2 + m\gamma)$ .

Let  $\pi : \mathfrak{h}_{\mathbb{R}}^* \rightarrow \mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2$  be the projection map given by

$$\pi(\mu) = \mu(H_1)\lambda_1 + \mu(H_2)\lambda_2. \quad (3.3)$$

Note that for  $H \in (\mathfrak{h}_{\mathcal{Fib}})_{\mathbb{R}}$ ,  $\pi(\mu)(H) = \mu(H)$ , and also,  $W_{\mathcal{Fib}}$  and  $\pi$  commute.

**Proposition 3.1.** *For  $m \in \mathbb{Z}$ ,  $\mathcal{Fib}(m)$  is an  $\mathfrak{h}_{\mathcal{Fib}}$ -diagonalizable, integrable  $\mathcal{Fib}$ -module where the action is the restriction to  $\mathcal{Fib}$  of the adjoint action of  $\mathcal{F}$ . In particular,*

- 1) the weights of  $\mathcal{Fib}(m)$  are  $P(\mathcal{Fib}(m)) = \pi(\Delta_m)$ ,
- 2) if  $\beta \in \Delta_m$  and  $x \in \mathcal{F}_{\beta}$ , then  $ad_H x = \pi(\beta)(H)x$  for  $H \in \mathfrak{h}_{\mathcal{Fib}}$ ,
- 3)  $\Delta_m$  and  $P(\mathcal{Fib}(m))$  are  $W_{\mathcal{Fib}}$ -invariant,
- 4)  $\pi(\beta_i^{\perp}) = \mathbb{R}\lambda_{3-i} = \beta_i^{\perp} \cap (\mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2)$  for  $i = 1, 2$ ,
- 5)  $\pi(P^{\pm}) \subset P_{\mathcal{Fib}}^{\pm}$ .

*Proof.* Let  $\beta = \begin{bmatrix} b+c+m & b \\ b & c \end{bmatrix} \in \Delta_m$  and  $x \in \mathcal{F}_\beta$ . Then

$$ad_{E_i}x = [E_i, x] \in \mathcal{F}_{\beta_i+\beta} \subset \mathcal{Fib}(m) \quad \text{and} \quad ad_{F_i}x = [F_i, x] \in \mathcal{F}_{-\beta_i+\beta} \subset \mathcal{Fib}(m),$$

and the action of each  $E_i$  and  $F_i$  for  $i = 1, 2$  is locally nilpotent on  $\mathcal{Fib}(m)$  since it is a multibracket of the Serre generators of  $\mathcal{F}$ , which have locally nilpotent action on all of  $\mathcal{F}$ . Moreover,  $\mathcal{Fib}(m)$  is  $\mathfrak{h}_{\mathcal{Fib}}$ -diagonalizable since for  $\beta \in \Delta_m$  and  $x \in \mathcal{F}_\beta$ ,

$$ad_Hx = [H, x] = \beta(H)x = \pi(\beta)(H)x \in \mathcal{F}_\beta \quad \text{for } H \in \mathfrak{h}_{\mathcal{Fib}},$$

which proves 1) and 2).

We have

$$r_1\beta = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b+c+m & b \\ b & c \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b+c+m & -2b-c-m \\ -2b-c-m & 3b+2c+m \end{bmatrix},$$

thus  $r_1\beta \in \Delta_m$ . Similarly,  $r_2\beta \in \Delta_m$ , since

$$r_2\beta = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b+c+m & b \\ b & c \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2c-b+m & c-b \\ c-b & c \end{bmatrix},$$

and since  $W_{\mathcal{Fib}}$  and  $\pi$  commute, we have  $W_{\mathcal{Fib}}P(\mathcal{Fib}(m)) = P(\mathcal{Fib}(m))$ , proving 3).

To show the projection map preserves the reflecting planes of  $W_{\mathcal{Fib}}$ , let  $\mu \in \beta_i^\perp$  for  $i = 1$  or  $2$ , so  $\mu(H_i) = 0$ . Then  $\pi(\mu) = \mu(H_1)\lambda_1 + \mu(H_2)\lambda_2 = \mu(H_{3-i})\lambda_{3-i}$  so  $\pi(\mu) \in \beta_i^\perp$ .

For 5), let  $\mu \in P^-$ . Then  $\pi(\mu) = \mu(H_1)\lambda_1 + \mu(H_2)\lambda_2 = \langle \beta_1, \mu \rangle \lambda_1 + \langle \beta_2, \mu \rangle \lambda_2$ . Since  $\langle \alpha_i, \mu \rangle = \frac{2(\alpha_i, \mu)}{(\alpha_i, \alpha_i)} = (\alpha_i, \mu) < 0$  it follows that  $\langle \beta_i, \mu \rangle = \frac{2(\beta_i, \mu)}{(\beta_i, \beta_i)} = (\beta_i, \mu) < 0$ . □

When viewing  $\mathcal{Fib}(m)$  as a  $\mathcal{Fib}$ -module, we write the weight-space decomposition as

$$\mathcal{Fib}(m) = \bigoplus_{\lambda \in P(\mathcal{Fib}(m))} \mathcal{Fib}(m)_\lambda. \quad (3.4)$$

We also have the multiplicity of  $\lambda$  in  $\mathcal{Fib}(m)$ ,

$$Mult_m(\lambda) = Mult_{\mathcal{Fib}(m)}(\lambda) = \dim_{\mathcal{F}}(\mathcal{F}_\beta), \quad (3.5)$$

where  $\beta \in \Delta_m$  is such that  $\pi(\beta) = \lambda$ .

**Definition 3.2.** If  $V^\lambda$  is a standard or non-standard  $\mathcal{Fib}$ -module with set of weights  $P(V^\lambda)$ , then for  $\mu \in P(V^\lambda)$ , the **inner multiplicity** of  $\mu$  in  $V^\lambda$  is

$$Mult_\lambda(\mu) = \dim_{V^\lambda}(V_\mu^\lambda).$$

### 3.2 Symmetries and cosets

The Cartan involution of  $\mathcal{Fib}$  is the restriction to  $\mathcal{Fib}$  of  $\nu : \mathcal{F} \rightarrow \mathcal{F}$  determined by  $\nu(e_i) = -f_i$ ,  $\nu(f_i) = -e_i$  for  $1 \leq i \leq 3$ , and  $\nu(h) = -h$  for  $h \in \mathfrak{h}$ . Since for  $\alpha \in \Phi$ ,  $\nu(\mathcal{F}_\alpha) = \mathcal{F}_{-\alpha}$  we also let  $\nu(\alpha) = -\alpha$ . Note that  $Mult_{\mathcal{F}}(\alpha) = Mult_{\mathcal{F}}(-\alpha)$ .

**Remark 3.3.** *It is sufficient to investigate the  $\mathcal{Fib}(m)$  decomposition for  $m \geq 0$ , since for all  $m \in \mathbb{Z}$ ,  $\nu(\mathcal{Fib}(m)) = \mathcal{Fib}(-m)$ . Moreover,*

- 1)  $V^\lambda \subset \mathcal{Fib}(m)$  is an irreducible HW  $\mathcal{Fib}$ -module with HWV  $v_\lambda$  if and only if  $V^{-\lambda} \subset \mathcal{Fib}(-m)$  is a irreducible LW  $\mathcal{Fib}$ -module with LWV  $\nu(v_\lambda) = v_{-\lambda}$ .
- 2) If  $V^\lambda \subset \mathcal{Fib}(m)$  is a non-standard irreducible  $\mathcal{Fib}$ -module with a generating vector  $v_\lambda$  of weight  $\lambda$ , then  $V^{-\lambda} \subset \mathcal{Fib}(-m)$  is a non-standard irreducible  $\mathcal{Fib}$ -module in  $\mathcal{Fib}(-m)$  with generating vector  $\nu(v_\lambda)$  of weight  $-\lambda$ .

The following lemma describes a symmetry in each  $\mathcal{Fib}$ -level that allows us to consider only positive weights in the decomposition of  $\mathcal{Fib}(m)$  for each  $m \in \mathbb{Z}$ .

**Lemma 3.4.** *Let  $\psi = w_1 w_3 \nu : \Phi \rightarrow \Phi$ . Then  $\psi$  is an involution of  $\Phi$  such that*

- 1)  $\psi(\Delta_m) = \Delta_m$ ,
- 2)  $\psi$  fixes  $\pm\alpha_1$  and  $\pm\alpha_3$ ,
- 3) If  $|m| \neq 1, 2$ ,  $\psi(\Delta_m \cap \Phi_+) = \Delta_m \cap \Phi_-$ ,
- 4)  $\psi(\Delta_1 \cap \Phi_+ \setminus \{\alpha_1\}) = \Delta_1 \cap \Phi_- \setminus \{-\alpha_1\}$ ,
- 5)  $\psi(\Delta_2 \cap \Phi_+ \setminus \{\alpha_3\}) = \Delta_2 \cap \Phi_- \setminus \{-\alpha_3\}$ ,
- 6)  $Mult_{\mathcal{F}}(\beta) = Mult_{\mathcal{F}}(\psi(\beta))$ ,
- 7)  $\psi$  commutes with  $\pi$ .

*Proof.* 1) We have  $M_{w_1 w_3} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ . Then

$$\psi(\beta) = - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -c & b \\ b & -a \end{bmatrix} \in \Delta_{a-b-c}.$$

In particular, if  $a = b + c + m$  then  $\psi(\beta) \in \Delta_m$  if  $\beta \in \Delta_m$ .

2 – 5) Also,  $\|\beta\| = ac - b^2 \geq -1 \Leftrightarrow ac \geq b^2 - 1$ , which gives us three cases:

I. If  $|b| \geq 2$ , then  $ac > 0$ , and we have  $\beta \in \Phi_+ \Leftrightarrow c < 0 \Leftrightarrow -a > 0 \Leftrightarrow \psi(\beta) \in \Phi_-$ .

II. If  $b = 0$ , then  $ac \geq -1$ . Assume  $ac = -1$ . Then  $\beta = \pm\alpha_3$ , and  $\psi(\pm\alpha_3) = w_1 w_3(\mp\alpha_3) = \pm\alpha_3$ . Now assume  $ac = 0$ . Then for some  $n > 0$ ,  $\beta = \pm n(\alpha_1 + \alpha_2) \in \Phi_{\pm}$  or  $\mp n(\alpha_2 + \alpha_2 + \alpha_3) \in \Phi_{\mp}$ , and an easy computation shows that  $\psi$  permutes these two sets of roots.

III. If  $|b| = 1$ , then  $ac \geq 0$ . If  $ac = 0$ ,  $\beta = \pm\alpha_1$ , and  $\psi(\beta) = w_1 w_3(\mp\alpha_1) = \pm\alpha_1$ . If  $ac > 0$  then  $\beta \in \Phi^{im}$ , and the argument from Case I still holds, so  $\beta \in \Phi_+ \Leftrightarrow \psi(\beta) \in \Phi_-$ .

6) follows from  $W$ - and  $\nu$ -invariance of  $Mult_{\mathcal{F}}(\cdot)$  and 7) follows because both  $W_{\mathcal{F}ib}$  and  $\nu$  commute with  $\pi$ .

□

**Remark 3.5.** For  $m \in \mathbb{Z}$ ,  $\psi$  corresponds to a reflection across the horizontal axis in the weight diagram for  $\mathcal{F}ib(m)$  (when viewed as a  $\mathcal{F}ib$ -module). Moreover for each weight  $\lambda \in \pi(\Delta_m)$ , we have

$$Mult_m(\lambda) = Mult_m(\psi(\lambda)).$$

Thus, when determining the decomposition of  $\mathcal{F}ib(m)$  into irreducible  $\mathcal{F}ib$ -modules, it is enough to consider only positive weights.

The  $\mathcal{F}ib$  root lattice  $Q_{\mathcal{F}ib}$  is an index 5 sublattice of  $P_{\mathcal{F}ib}$ , so the quotient module  $P_{\mathcal{F}ib}/Q_{\mathcal{F}ib}$  consists of 5 cosets. We denote the  $i^{th}$  coset of  $P_{\mathcal{F}ib}/Q_{\mathcal{F}ib}$  by  $K_i$  where  $i \in \{-2, -1, 0, 1, 2\}$ . Then

$$K_i = \{\Lambda_i + n_1\beta_1 + n_2\beta_2 \mid n_1, n_2 \in \mathbb{Z}\}, \quad (3.6)$$

where the representative weight  $\Lambda_i \in \pi(\Delta_i)$  is from Table 3.1.

$m$ (level)	$\Lambda_m$	weight	projection
-2	$\Lambda_{-2}$	$-(\lambda_1 - \lambda_2)$	$= \pi(-\alpha_3)$
-1	$\Lambda_{-1}$	$2(\lambda_1 - \lambda_2)$	$= \pi(\alpha_1)$
0	$\Lambda_0$	0	0
1	$\Lambda_1$	$-2(\lambda_1 - \lambda_2)$	$= \pi(-\alpha_1)$
2	$\Lambda_2$	$\lambda_1 - \lambda_2$	$= \pi(\alpha_3)$

Table 3.1: Coset representative  $\Lambda_m = \pi(\beta)$  for some  $\beta \in \Delta_m^{re}$ ,  $|m| \leq 2$ .

Each root of  $\Phi$  projects to a weight in one of the five cosets  $K_i$  (each invariant under  $W_{\mathcal{F}ib}$ ), and all roots from the same level project to weights in the same coset. In proposition 3.6 we use  $a \bmod 5$  to mean the unique integer  $b \in \{0, 1, 2, 3, 4\}$  such that  $b \equiv a \bmod 5$ .

**Proposition 3.6.** Let  $m \in \mathbb{Z}$ . If  $\beta \in \Delta_m$ , then  $\pi(\beta) \in K_{(m+2)(\bmod 5)-2}$ .

*Proof.* We have for any  $m \in \mathbb{Z}$ ,

$$\beta = \begin{bmatrix} b+c+m & b \\ b & c \end{bmatrix} = \begin{bmatrix} b+c & b \\ b & c \end{bmatrix} + \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix} = \beta' + m\lambda_1, \quad \text{where } \beta' = \begin{bmatrix} b+c & b \\ b & c \end{bmatrix} \in \Delta,$$

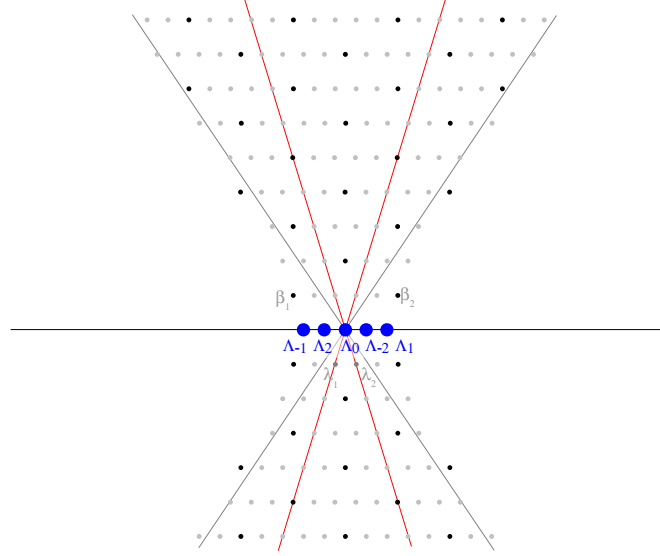


Figure 3.2:  $Q_{\mathcal{Fib}}$  is an index 5 sublattice of  $P_{\mathcal{Fib}}$  in the  $\mathcal{Fib}$ -plane. This figure shows only those lattice points corresponding to real or imaginary weights of  $\mathcal{Fib}$ .

giving the projection  $\pi(\beta) = \beta' + m\lambda_1 \in Q_{\mathcal{Fib}} + m\lambda_1$ . Thus  $\pi(\beta)$  and  $m\lambda_1$  lie in the same coset  $K_{i(m)}$ , where  $i(m) \in \{-2, -1, 0, 1, 2\}$ . Note also that  $K_{i(m)}$  is invariant under addition by  $\mathbb{Z}$ -linear combinations of the simple roots, so that

$$m\lambda_1 + m\beta_2 = -2m(\lambda_1 - \lambda_2)$$

also lies in  $K_{i(m)}$ . Using Table 3.1, we have  $i(m) = (m + 2)(\text{mod } 5) - 2$ .  $\square$

It can be seen that the projection map partitions the levels with respect to  $\mathcal{Fib}$ . Let  $\mathfrak{L} = \{\Delta_m \mid m \in \mathbb{Z}\}$ . For  $|m| \leq 2$ , the pullback  $\pi^{-1}(K_m) = \{\Delta_j \mid j \equiv m(\text{mod } 5)\}$  is an equivalence class of levels, giving the partition  $\mathfrak{L} = \bigsqcup_{|m| \leq 2} \pi^{-1}(K_m)$ .

### 3.3 Is $\mathcal{Fib}(m)$ completely reducible for all $m \in \mathbb{Z}$ ?

We now return to the investigation of the decomposition of  $\mathcal{F}$  into a direct sum of irreducible  $\mathcal{Fib}$ -modules. Recall that in [FF], it was shown that  $\mathcal{A}ff(0)$  was the single irreducible adjoint representation, and each  $\mathcal{A}ff(m)$  for  $m \neq 0$  was a direct sum of modules from either category  $\mathcal{O}$  (if  $m > 0$ ) or  $\mathcal{O}^{op}$  (if  $m < 0$ ), and so was completely reducible by Theorem 1.34. It is therefore not unreasonable to conjecture the following for the modules  $\mathcal{Fib}(m)$ :

**Conjecture 3.7.** *For each  $m \in \mathbb{Z}$ ,  $\mathcal{Fib}(m)$  completely reduces into a direct sum of one trivial module (on level 0), one non-standard module (on levels  $|m| \leq 2$ ), and standard modules (highest and lowest) on all levels.*

**Proposition 3.8.** *The only trivial  $\mathcal{Fib}$ -modules in  $\mathcal{F}$  are in  $\mathfrak{h}_{\mathcal{F}}$ .*

*Proof.* Let  $V$  be a trivial  $\mathcal{Fib}$ -module with set of weights  $P(V)$  and weight-space decomposition

$$V = \bigoplus_{\lambda \in P(V)} V_{\lambda}.$$

Let  $\alpha = \sum_{i=1}^3 n_i \alpha_i \in \Phi \cup \{0\}$  such that  $\pi(\alpha) = \lambda \in P(V)$ . We will show that  $\alpha$  is necessarily 0, and therefore  $P(V) = \{0\}$  and  $V \subset \mathfrak{h}_{\mathcal{F}}$ .

For all  $X_{\alpha} \in \mathcal{F}_{\alpha}$ , we have  $[H_i, X_{\alpha}] = \alpha(H_i)X_{\alpha} = 0$  for  $i = 1, 2$ . This gives us

$$\begin{cases} -2n_1 + 2n_2 - n_3 = 0 \\ 2n_1 - 3n_2 + n_3 = 0 \end{cases},$$

which has solutions  $\alpha = c(\alpha_1 - 2\alpha_3)$  where  $c \in \mathbb{Z}$ . However, if  $c \neq 0$  then  $\alpha \notin \Phi$ , therefore  $\alpha$  must be 0.  $\square$

The following two lemmas will help to prove that for  $|m| > 2$ ,  $\mathcal{Fib}(m)$  is a direct sum of irreducible standard modules from both category  $\mathcal{O}$  and  $\mathcal{O}^{op}$ . The cases of  $|m| \leq 2$  are not as clear, and will be explored afterwards.

**Lemma 3.9.** *If  $\beta \in \Delta_m$  for  $m \in \mathbb{Z}$ , then  $\|\pi(\beta)\|^2 = \|\beta\|^2 - \frac{2m^2}{5}$ .*

*Proof.* Write  $\beta = \begin{bmatrix} b+c+m & b \\ b & c \end{bmatrix} = \begin{bmatrix} b+c & b \\ b & c \end{bmatrix} + \begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}$ , so that  $\beta = (-c)\beta_1 + (-b-c)\beta_2 + m(-\alpha_1 - \alpha_2)$ . Then the projection

$$\pi(\beta) = \beta(H_1)\lambda_1 + \beta(H_2)\lambda_2 = (3b+c+m)\lambda_1 + (c-2b)\lambda_2,$$

and

$$\begin{aligned} \|\pi(\beta)\|^2 &= \langle (3b+c+m)\lambda_1 + (c-2b)\lambda_2, (3b+c+m)\lambda_1 + (c-2b)\lambda_2 \rangle \\ &= -\frac{2}{5} \left[ (3b+c+m)^2 + (c-2b)^2 \right] - 2\left(\frac{3}{5}\right)(3b+c+m)(c-2b) \\ &= 2b^2 - 2c^2 - 2(b+m)c - \frac{2m^2}{5} = \|\beta\|^2 - \frac{2m^2}{5}. \end{aligned}$$

$\square$

This immediately gives the following.

**Lemma 3.10.** *We have:*

a) *If  $|m| > 2$  and  $\beta \in \Delta_m$ , then  $\|\pi(\beta)\|^2 < 0$ . In other words,  $\pi(\Delta_m) \subset LC_{\mathcal{Fib}}$ .*

b) *If  $|m| \leq 2$  and  $\beta \in \Delta_m$ , then  $\|\pi(\beta)\|^2 > 0$  if and only if  $\beta$  is real ( $\|\beta\|^2 = 2$ ).*

**Proposition 3.11.** *If  $|m| > 2$ , then  $\mathcal{Fib}(m)$  is a sum of integrable highest-weight  $\mathcal{Fib}$ -modules in category  $\mathcal{O}$  (which are therefore completely reducible) and integrable lowest-weight  $\mathcal{Fib}$ -modules in category  $\mathcal{O}^{op}$  (which are also completely reducible), hence  $\mathcal{Fib}(m)$  is completely reducible.*

*Proof.* We have the set of weights  $P(\mathcal{Fib}(m)) = \pi(\Delta_m) = P(\mathcal{Fib}(m))_- \cup P(\mathcal{Fib}(m))_+$ , and we consider  $\mathcal{Fib}(m)_\pm$  (as in Definition 1.41). By Lemma 3.10, if  $|m| > 2$  then  $P(\mathcal{Fib}(m))$  contains only imaginary weights inside the light-cone on level 0, so  $P(\mathcal{Fib}(m))_\pm \subset LC_\mp$ . Furthermore, since  $W_{\mathcal{Fib}}(LC_+) \cap W_{\mathcal{Fib}}(LC_-) = \{0\}$ , we have that  $\mathcal{Fib}(m)_+$  and  $\mathcal{Fib}(m)_-$  are integrable  $\mathcal{Fib}$ -modules from category  $\mathcal{O}$  and  $\mathcal{O}^{op}$ , respectively. By Theorem 1.34 and Remark 3.3,  $\mathcal{Fib}(m)_+$  (resp.  $\mathcal{Fib}(m)_-$ ) completely reduces as a direct sum of irreducible highest-weight (resp. lowest-weight)  $\mathcal{Fib}$ -modules. Thus,  $\mathcal{Fib}(m) = \mathcal{Fib}(m)_- \oplus \mathcal{Fib}(m)_+$  is completely reducible.  $\square$

**Lemma 3.12.** *If  $V \subseteq \mathcal{Fib}(m)$  is a standard highest-weight  $\mathcal{Fib}$ -module, then  $P(V) \subset LC_{\mathcal{Fib},-}$ . If  $V \subseteq \mathcal{Fib}(m)$  is a standard lowest-weight  $\mathcal{Fib}$ -module, then  $P(V) \subset LC_{\mathcal{Fib},+}$ . If  $V \subseteq \mathcal{Fib}(m)$  is a non-standard module, then a  $\lambda \in P(V)$  satisfying Definition 1.42 (ii) has positive norm, and  $W_{\mathcal{Fib}}\lambda \subset P(V) \subseteq \{\mu \in K_m \mid \|\mu\| \leq \|\lambda\|\}$ .*

*Proof.* Let  $m \in \mathbb{Z}$  and let  $V \subseteq \mathcal{Fib}(m)$  be a standard module. Assume  $V$  is a highest-weight module with highest weight  $\lambda \in P_{\mathcal{Fib}}^+$ , and consider  $\lambda \geq \mu \in P(V)$ . Let  $\mu' \in P_{\mathcal{Fib}}^+$  be a  $W_{\mathcal{Fib}}$ -conjugate of  $\mu$ . Then since  $\mu' \leq \lambda$  and  $\mu, \mu'$  lie on a branch of the hyperbola  $S_{\mathcal{Fib}, \|\mu\|^2}$  that lies below the hyperbola containing  $\lambda$  (cf. Figure 2.2), we have that  $\|\mu\|^2 \leq \|\lambda\|^2 < 0$ . Since  $P(V)$  is  $W_{\mathcal{Fib}}$ -invariant and  $W_{\mathcal{Fib}}$  preserves squared-length,  $P(V) \subset LC_{\mathcal{Fib},+}$ . Similarly, if  $V$  is a lowest-weight module with lowest weight  $\lambda \in P_{\mathcal{Fib}}^-$ , then  $P(V) \subset LC_{\mathcal{Fib},-}$ .

Assume  $V$  is a non-standard module but that  $P(V) \subset LC_{\mathcal{Fib}}$ . Then by Definition 1.42 (ii) there exists  $\lambda \in P(V)$  such that  $\lambda \notin P_{\mathcal{Fib}}^\pm$ ,  $\|\lambda\|$  is maximal in  $\{\|\mu\| \mid \mu \in P(V)\}$ ,  $|wt(\lambda)|$  is minimal among those, and  $V = \mathcal{U}(\mathcal{Fib}) \cdot v_\lambda$  for  $0 \neq v_\lambda \in V_\lambda$ . Then  $\lambda$  is  $W_{\mathcal{Fib}}$ -conjugate to a unique weight  $\mu \in P_{\mathcal{Fib}}^\pm$ , and since  $V$  is irreducible,  $V = \mathcal{U}(\mathcal{Fib}) \cdot v_\mu$  for any  $0 \neq v_\mu \in V_\mu$ . Moreover, since  $\lambda, \mu$  lie on the same branch of a hyperbola inside the light-cone of constant squared-length  $\|\lambda\|^2$ , we have  $|wt(\mu)| < |wt(\lambda)|$ , contradicting the minimality of  $|wt(\lambda)|$ . Therefore,  $P(V)$  must contain weights of positive norm, including any  $\lambda$  from Definition 1.42 (ii). Since  $P(V)$  is  $W_{\mathcal{Fib}}$ -invariant and saturated (cf. Definition 1.33), for any  $w \in W_{\mathcal{Fib}}$ ,  $\beta \in \Delta$ ,  $S_\beta(w\lambda)$  contains weights inside  $LC_{\mathcal{Fib}}$ . So  $P(V) \subseteq \{\mu \in K_m \mid \|\mu\| \leq \|\lambda\|\}$ .  $\square$

**Lemma 3.13.** *We have*

$$i) \quad \pi(\Delta_0^{re}) = \Delta^{re} = W_{\mathcal{Fib}}\beta_1 \sqcup W_{\mathcal{Fib}}\beta_2,$$

$$ii) \quad \text{If } |m| = 1, 2, \quad \pi(\Delta_m^{re}) = W_{\mathcal{Fib}}(\Lambda_m).$$

*Proof.* Let  $|m| \leq 2$ . We have the “real” hyperbola  $\pi(S_2 \cap \mathbb{R}\beta_1 \oplus \mathbb{R}\beta_2 + m\gamma) = S(m)^1 \cup S(m)^2$ , where  $S(m)^i$  for  $i = 1, 2$  is a branch of a hyperbola defined by

$$S(m)^i = \left\{ \mu \in (\mathfrak{h}_{\mathcal{Fib}}^*)_{\mathbb{R}} \mid \|\mu\|^2 = 2 - \frac{2m^2}{5}, \quad \langle \mu, \beta_i \rangle > 0 \quad \text{and} \quad \langle \mu, \beta_{3-i} \rangle < 0 \right\}.$$

(In all weight diagrams in the present work,  $S(m)^1$  is on the left and  $S(m)^2$  is on the right.) If  $\beta \in \Delta_m^{re}$ , then  $\pi(\beta) \in S(m)^i$  for  $i = 1$  or  $i = 2$ .

First we make the following general observations. If  $\lambda = c_1\lambda_1 + c_2\lambda_2 \in S(m)^i$ , then  $c_i > 0$ ,  $c_{3-i} < 0$  and

$$r_i\lambda = \lambda - \langle \lambda, \beta_i \rangle \beta_i = c_1\lambda_1 + c_2\lambda_2 - c_i(2\lambda_i - 3\lambda_{3-i}),$$

and since  $wt(r_i\lambda) = c_1 + c_2 + c_i$  and  $wt(\lambda) = c_1 + c_2$ , we have

$$wt(r_i\lambda) - wt(\lambda) = c_i > 0 \quad \text{and} \quad wt(r_{3-i}\lambda) - wt(\lambda) = c_{3-i} < 0.$$

This gives us

$$\lambda \in S(m)^i \quad \Rightarrow \quad wt(r_i\lambda) > wt(\lambda) \quad \text{and} \quad wt(r_{3-i}\lambda) < wt(\lambda). \quad (3.7)$$

Note that  $\pi(\Delta_m)$  partitions as

$$\pi(\Delta_m) = P(m)_- \cup \{\Lambda_m\} \cup P(m)_+,$$

where

$$P(m)_- = \{\mu \in \pi(\Delta_m) \mid wt(\mu) < 0\} \quad \text{and} \quad P(m)_+ = \{\mu \in \pi(\Delta_m) \mid wt(\mu) > 0\}.$$

Let  $P(m)_{\pm}^{re} = P(m)_{\pm} \cap \pi(\Delta_m^{re})$ , and let  $\beta \in P(m)_-^{re}$  (by  $\psi$ -symmetry, it is enough to consider only negative weights). The set  $W_{\mathcal{Fib}}\pi(\beta) \cap P(m)_-^{re}$  is graded by  $wt$ , is bounded above by 0, so it contains an element  $\lambda(m)^i = d_1\lambda_1 + d_2\lambda_2 \in S(m)^i$  for  $i = 1$  or  $i = 2$  such that  $wt(\lambda(m)^i) = d_1 + d_2 < 0$  is maximal. By the implications (3.7), we have  $wt(r_i\lambda(m)^i) > wt(\lambda(m)^i)$ . But  $wt(\lambda(m)^i)$  is maximal in  $W_{\mathcal{Fib}}\pi(\beta) \cap P(m)_-^{re}$ , so if  $m \neq 0$ , then either  $r_i\lambda(m)^i \in P(m)_+^{re}$  or  $r_i\lambda(m)^i = \Lambda_m$ , and if  $m = 0$ , then  $r_i\lambda(0)^i \in P(0)_+^{re}$  (since  $\Lambda_0 = 0$  is not a  $W_{\mathcal{Fib}}$ -conjugate of the projection of a real root).

Each of the following weights  $\mu(m)^i \in S(m)^i \cap P(m)_-^{re}$  satisfies  $0 > wt(\mu(m)^i) \geq -2$ , and is of maximal  $wt$  in  $P(m)_-^{re}$  (see Figure 3.2):

$$\begin{aligned} \mu(0)^1 &= 2\lambda_1 - 3\lambda_2 = \beta_1, & \mu(1)^1 &= \pi(-\alpha_1) = 2\lambda_1 - 4\lambda_2, & \mu(2)^2 &= \pi(\alpha_3) = -2\lambda_1 + \lambda_2, \\ \mu(0)^2 &= -3\lambda_1 - 2\lambda_2 = \beta_2, & \mu(-1)^2 &= \pi(\alpha_1) = -4\lambda_1 + 2\lambda_2, & \mu(-2)^1 &= \pi(-\alpha_3) = \lambda_1 - 2\lambda_2. \end{aligned}$$

Then  $\mu(m)^i$  is a candidate for  $\lambda(m)^i$ . We see that if  $m = 0$ , then  $\pi(\beta)$  is a  $W_{\mathcal{Fib}}$ -conjugate of either  $\beta_1$  or  $\beta_2$ , and if  $m \neq 0$  we have:



- if  $m = -2$ , then  $r_1(\mu(-2)^1) = r_1(\lambda_1 - 2\lambda_2) = -\lambda_1 + \lambda_2 = \Lambda_{-2}$ ,
- if  $m = -1$ , then  $r_2(\mu(-1)^2) = r_2(-4\lambda_1 + 2\lambda_2) = 2\lambda_1 - 2\lambda_2 = \Lambda_{-1}$ ,
- if  $m = 1$ , then  $r_1(\mu(1)^1) = r_1(2\lambda_1 - 4\lambda_2) = -2\lambda_1 + 2\lambda_2 = \Lambda_1$ ,
- if  $m = 2$ , then  $r_2(\mu(2)^2) = r_2(-2\lambda_1 + \lambda_2) = \lambda_1 - \lambda_2 = \Lambda_2$ ,

completing the proof.  $\square$

**Theorem 3.14.** *Let  $|m| \leq 2$ , and let  $V \subseteq \mathcal{Fib}(m)$  be a  $\mathcal{Fib}$ -submodule. Then the following are equivalent:*

- i)  $\pi(\beta) \in P(V)$  for some  $\beta \in \Delta_m^{re}$ ,
- ii)  $\Lambda_m \in P(V)$  and  $V$  is non-trivial,
- iii)  $\pi(\Delta_m^{re}) \subset P(V)$ ,
- iv)  $P(V) = P(\mathcal{Fib}(m))$ .

*Proof.* Assume  $\pi(\beta) \in P(V)$  for some  $\beta \in \Delta_m^{re}$ . By Proposition 3.8,  $P(V) \neq \{0\}$  so  $V$  is not a trivial module. If  $|m| = 1, 2$ , then  $\Lambda_m \in W_{\mathcal{Fib}}\pi(\beta) \subset P(V)$  by Lemma 3.13(ii) and the fact that  $P(V)$  is  $W_{\mathcal{Fib}}$ -invariant. If  $m = 0$ , then  $\beta_i \in W_{\mathcal{Fib}}\pi(\beta)$  for  $i = 1$  or  $i = 2$  by Lemma 3.13(i). Since  $P(V)$  is a saturated set, we have  $S_{\beta_i}(\beta_i) = \{\beta_i, \Lambda_0, -\beta_i\} \subset P(V)$  (cf. (1.3)), so (i)  $\Rightarrow$  (ii).

If  $|m| = 1, 2$ , then  $\pi(\Delta_m^{re}) = W_{\mathcal{Fib}}\Lambda_m \subset P(V)$  by part (ii) and Lemma 3.13(ii). If  $m = 0$ , we have that  $P(V)$  is a saturated set that contains  $\Lambda_0$ , and since  $V$  is not trivial,  $S_{\beta_i}(\beta_i) \subset P(V)$  for  $i = 1$  and  $i = 2$ , so  $P(V)^{re} = W_{\mathcal{Fib}}\{\beta_1\} \sqcup W_{\mathcal{Fib}}\{\beta_2\} = \pi(\Delta^{re}) \subset P(V)$ , so (ii)  $\Rightarrow$  (iii).

Thus  $P(\mathcal{Fib}(m))$  and  $P(V)$  contain the same set of maximal-norm weights  $\pi(\Delta_m^{re})$ , and each is a saturated,  $W_{\mathcal{Fib}}$ -invariant set. Therefore, if weight strings  $S_{\beta_i}(\mu) \subset P(\mathcal{Fib}(m))$  for some  $\mu \in \pi(\Delta_m)$  and  $i = 1, 2$ , then also  $S_{\beta_i}(\mu) \subset P(V)$ . Thus  $P(\mathcal{Fib}(m)) = P(V)$ , so (iii)  $\Rightarrow$  (iv).

Lastly, (iv)  $\Rightarrow$  (i) is obviously true, so all four conditions are equivalent.  $\square$

If  $|m| \leq 2$ , then Lemma 3.10 shows that all real roots of  $\Delta_m$  project outside the  $\mathcal{Fib}$  light-cone. Since  $\mathcal{Fib}(m)$  is a  $\mathcal{Fib}$ -module, these projected weights are part of a  $\mathcal{Fib}$ -module which is neither in category  $\mathcal{O}$  nor in  $\mathcal{O}^{op}$ . A theorem analogous to Theorem 1.34 on the complete reducibility of general integrable modules has not been found in the literature. The irreducible adjoint representation for  $\mathcal{Fib}$  is integrable but neither in category  $\mathcal{O}$  nor  $\mathcal{O}^{op}$ . We write this non-standard  $\mathcal{Fib}$ -module as  $V^{\pi(\beta_i)}$  for either  $i = 1$  or  $i = 2$ . The next results will show that  $\mathcal{Fib}$  is the only non-standard module on level 0.

**Theorem 3.15.** *Let  $|m| \leq 2$ . Then  $\mathcal{Fib}(m)$  has a unique irreducible non-standard quotient whose set of weights is the same as  $P(\mathcal{Fib}(m))$ . For the case  $m = 0$ , this quotient is the adjoint representation of  $\mathcal{Fib}$ .*

*Proof.* Let  $|m| \leq 2$ , and define  $V(m) = \mathcal{U}(\mathcal{Fib}) \cdot v_\beta$  where  $\beta$  and  $v_\beta$  are chosen from Table 3.2. Since  $\beta$  is a real root, we have that  $Mult_{\mathcal{F}}(\beta) = 1$ ,  $(v_\beta)$  is a basis for  $\mathcal{F}_\beta$ , and by Theorem 3.14,  $P(V(m)) = P(\mathcal{Fib}(m))$ . Also,  $\pi(\beta) \notin LC_{\mathcal{Fib}}$  so, by Lemma 3.12,  $V(m)$  is not a sum

$m$ (level)	$\beta \in \Delta_m^{re}$	$\pi(\beta)$	$v_\beta \in \mathcal{F}_\beta$
-2	$-\alpha_3$	$\Lambda_{-2}$	$f_3$
-1	$\alpha_1$	$\Lambda_{-1}$	$e_1$
0	$\beta_2$	$\beta_2$	$E_2 = e_2$
1	$-\alpha_1$	$\Lambda_1$	$f_1$
2	$\alpha_3$	$\Lambda_2$	$e_3$

Table 3.2: A choice of real root  $\beta \in \Delta_m^{re}$  for  $|m| \leq 2$  and a vector  $v_\beta \in \mathcal{F}_\beta$  that generates the  $\mathcal{Fib}$ -module  $V(m) = \mathcal{U}(\mathcal{Fib}) \cdot v_\beta$ .

of standard modules.

By Borchers, there exists a maximal proper  $\mathcal{Fib}$ -submodule  $U(m) \subset V(m)$  not containing  $v_\beta$ , and the quotient  $V(m)/U(m)$  is irreducible ([B]). Since  $V(m)/U(m) = \mathcal{U}(\mathcal{Fib}) \cdot \overline{v_\beta}$  and  $\pi(\beta)$  meets the criteria in Definition 1.42 (ii), this quotient module is non-standard, and we may write it as  $V^{\beta_2}$  if  $m = 0$ , as  $V^{\Lambda_m}$  if  $|m| = 1, 2$ , or more generally as  $V^{\pi(\beta)}$ .

Let  $Y(m) = \mathcal{Fib}(m)/V(m)$ . Then for each  $\mu \in \pi(\Delta_m)$ ,

$$Mult_m(\mu) = Mult_{V(m)}(\mu) + Mult_{Y(m)}(\mu).$$

Since  $Mult_m(\pi(\beta)) = Mult_{V(m)}(\pi(\beta)) = 1$ , we have  $Mult_{Y(m)}(\pi(\beta)) = 0$ . Thus  $\pi(\beta) \notin P(Y(m))$ . By Theorem 3.14,  $P(Y(m)) \cap \pi(\Delta_m^{re}) = \emptyset$ , hence  $P(Y(m)) \subset LC_{\mathcal{Fib}}$ , so  $Y(m)$  cannot contain any other non-standard modules. Therefore  $V^{\Lambda_m}$  is the unique irreducible non-standard module in  $\mathcal{Fib}(m)$ .

In the case  $m = 0$  we have  $V(0) = \mathcal{U}(\mathcal{Fib}) \cdot E_2 = \mathcal{Fib}$ , and since the adjoint representation of  $\mathcal{Fib}$  contains no proper submodules, we have that  $U(0) = \{0\}$ , hence  $V(m) = \mathcal{Fib}$ .  $\square$

**Proposition 3.16.** *For  $0 < |m| \leq 2$ , let  $Y(m) = \mathcal{Fib}(m)/V(m)$  as in Proposition 3.15. Then*

- i)  $P(Y(m))$  consists only of weights  $\mu$  where  $\|\mu\|^2 < 0$ , and
- ii)  $Y(m)$  is a sum of integrable highest-weight  $\mathcal{Fib}$ -modules in category  $\mathcal{O}$  (which are therefore completely reducible) and integrable lowest-weight  $\mathcal{Fib}$ -modules in category  $\mathcal{O}^{op}$  (which are also completely reducible), hence  $Y(m)$  is completely reducible.

*Proof.* (i): Let  $0 < |m| \leq 2$ . Then for all  $\beta \in \Delta_m^{re}$ , we have  $Mult_{\mathcal{F}}(\beta) = 1$ , so

$$Mult_{Y(m)}(\pi(\beta)) = Mult_{\mathcal{F}ib(m)}(\pi(\beta)) - Mult_{V(m)}(\pi(\beta)) = 0,$$

so  $\pi(\beta) \notin P(Y(m))$ . By Lemma 3.10,  $P(Y(m))$  contains only imaginary weights that are projections of imaginary roots in  $\Delta_m$ .

(ii): Since  $P(Y(m)) = \pi(\Delta_m^{im})$  lies strictly inside the light-cone  $LC_{\mathcal{F}ib}$  for  $|m| = 1, 2$ , we have  $P(Y(m))_{\pm} \subset LC_{\mathcal{F}ib, \mp}$  (cf. Definition 1.41), and  $P(Y(m))_- = \psi(P(Y(m))_+)$ .

Since  $W_{\mathcal{F}ib}(LC_{\mathcal{F}ib,+}) \cap W_{\mathcal{F}ib}(LC_{\mathcal{F}ib,-}) = \{0\}$ , we have that  $Y(m)_+$  and  $Y(m)_-$  are integrable  $\mathcal{F}ib$ -modules from category  $\mathcal{O}$  and  $\mathcal{O}^{op}$ , respectively, and so by Theorem 1.34, are completely reducible. Thus,  $Y(m) = Y(m)_- \oplus Y(m)_+$  is also completely reducible.  $\square$

The case  $m = 0$  (excluded from the proposition) is slightly complicated by the presence of a trivial module  $V^0$  for  $\mathcal{F}ib$  on level 0. In Chapter 4, we will prove the existence of  $V^0$ , and we will then show that

$$\mathcal{F}ib(0) = V^0 \oplus \mathcal{F}ib \oplus Y(0)$$

is completely reducible. This follows from the fact that  $V(m) = \mathcal{F}ib$  is simple, so  $U(0) = \{0\}$ . However, for  $|m| = 1, 2$ ,  $U(m)$  is not necessarily trivial. The third isomorphism theorem gives us

$$(\mathcal{F}ib(m)/U(m)) / (V(m)/U(m)) \cong \mathcal{F}ib(m)/V(m),$$

and since  $V(m)/U(m) = V^{\Lambda_m}$  is irreducible, it is a direct summand of  $\mathcal{F}ib(m)/U(m)$ , so knowledge of the inner multiplicities of  $V^{\Lambda_m}$  will give outer multiplicities of other irreducible modules in the decomposition of  $\mathcal{F}ib(m)/U(m)$ , not  $\mathcal{F}ib(m)$  as desired. In the present work we assume the following conjecture is true.

**Conjecture 3.17.** *For  $0 < |m| \leq 2$ , the maximal submodule  $U(m)$  from Theorem 3.15 is  $\{0\}$ , and  $V(m) = V^{\Lambda_m}$  is a direct summand of  $\mathcal{F}ib(m)$ , so  $\mathcal{F}ib(m) = V^{\Lambda_m} \oplus Y(m)_- \oplus Y(m)_+$ .*

Note that if this conjecture can be proven true, then this will complete the proof of Conjecture 3.7.

### 3.4 Determining inner multiplicities of irreducible submodules

We now outline procedures for determining the inner multiplicities of any irreducible  $\mathcal{F}ib$ -module  $U$ . Since  $\nu(\mathcal{F}ib(m)) = \mathcal{F}ib(-m)$  for all  $m \in \mathbb{Z}$ , we may assume  $m \geq 0$ , and let  $U \subset \mathcal{F}ib(m)$  be an irreducible  $\mathcal{F}ib$ -submodule. The Weyl subgroup  $W_{\mathcal{F}ib}$  preserves  $P(U)$ , and for all  $\mu \in P(U)$ ,  $Mult_U(w\mu) = Mult_U(\mu)$  for each  $w \in W_{\mathcal{F}ib}$ . Thus, determining the inner multiplicities of weights in  $P(U)$  reduces to calculating multiplicities of only those weights inside the fundamental domain.

If  $U = V^\lambda$  is a standard lowest-weight module with lowest weight  $\lambda \in P^-$ , the weight diagram  $P(U)$  labeled with inner multiplicities can be found in the following recursive way. For an example, we refer the reader to 4.2b in Chapter 4. First, we determine the weights of  $P(U)$ . Starting with  $\lambda$ , find all weight strings that result from simple  $W_{\mathcal{F}ib}$  reflections of  $\lambda$  (by reflecting across the lines  $\mathbb{R}\lambda_1, \mathbb{R}\lambda_2$ ), and add all the weights from those weight strings to the diagram. Then, reflect all of the previously found weights across each reflecting line, add those weight strings to the diagram, and repeat. Next we find the inner multiplicities of weights in the diagram, which follow the Racah-Speiser recursion,

$$Mult_\lambda(\mu) = - \sum_{1 \neq w \in W_{\mathcal{F}ib}} \det(w) Mult_\lambda(\mu + (w\rho - \rho))$$

where  $\rho = \lambda_1 + \lambda_2$ . This author used the geometric approach outlined by Feingold in [F3], which we now describe for  $U = V^\lambda$ . The weights of  $P(U)$  are partially ordered by height, whereby  $\mu > \lambda$  if and only if  $ht(\mu - \lambda) > 0$ . We start with the fact that for the lowest weight  $\lambda$ ,  $Mult_\lambda(\lambda) = 1$ . Now let  $\lambda < \mu \in P(U)$ , and assume that for all  $\lambda \leq \gamma < \mu$ ,  $Mult_\lambda(\gamma)$  is already known. Refer to Figure 3.3a for a diagram showing the  $W_{\mathcal{F}ib}$  conjugates of  $\rho$ . For all  $1 \neq w \in W_{\mathcal{F}ib}$ , note that  $w\rho - \rho$  is in  $\Phi_{\mathcal{F}ib,-}$ . The Racah-Speiser recursion states that the multiplicity of  $\mu$  is a sum of signed multiplicities of weights in the diagram of  $P(U)$  lower than  $\mu$ , where the sign is determined by  $-\det(w) = (-1)^{\ell(w)+1}$  where  $\ell : W_{\mathcal{F}ib} \rightarrow \mathbb{Z}$  is the standard length function on  $W_{\mathcal{F}ib}$  with respect to  $r_1, r_2$ . Feingold noted that one may

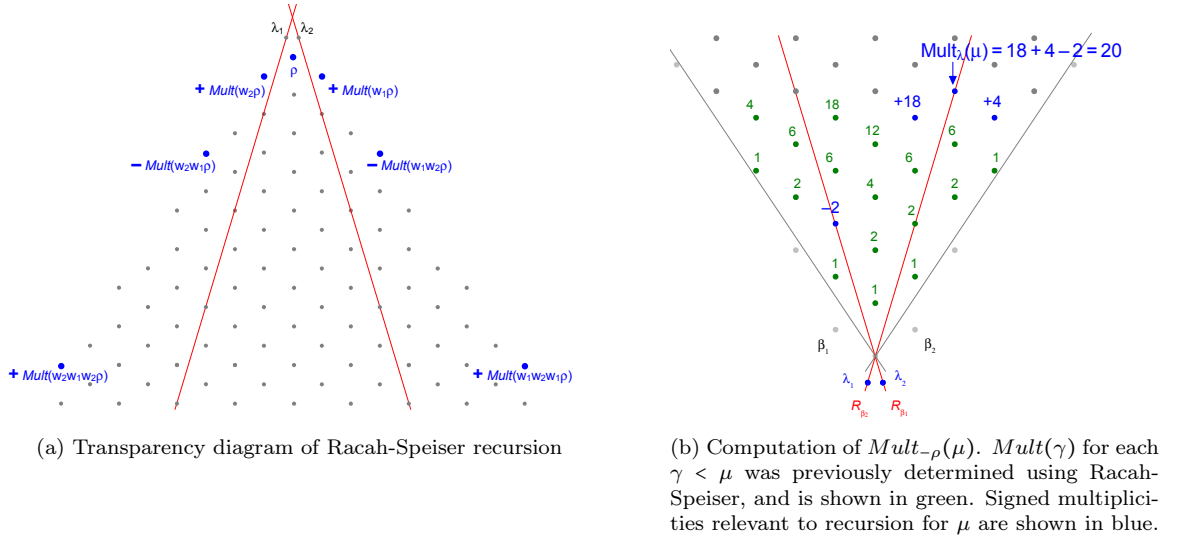


Figure 3.3: Explanation of Racah-Speiser recursion for the example of  $V^{-\rho} \subset \mathcal{F}ib(0)$ . The red lines are the reflecting lines  $\beta_1^\perp, \beta_2^\perp$ .

make a transparency of the  $W_{\mathcal{F}ib}$ -conjugates of  $\rho$  (with the same lattice spacing as  $P(U)$ ), label it with signs to denote which multiplicities get added or subtracted, and overlay this

transparency onto the weight diagram of  $U$ , being sure to line up  $\mu$  with  $\rho$ . Then, one may quickly compute  $Mult_\lambda(\mu)$  by adding the signed multiplicities of weights referred to in the Racah-Speiser formula. See Figure 3.3b for an example (note that Figure 4.2b shows the completed diagram up to height 12).

If  $U = \mathcal{Fib}$  is the adjoint representation on level 0, the weight multiplicities follow a Kac-Peterson recursion and have already been determined by Kac (Chapter 11 of [K2]).

If  $U$  is a non-standard module on levels  $\pm 1$  or  $\pm 2$ , there is no recursion found in the literature for determining the weight multiplicities of  $U$ . The approach for finding these multiplicities (which was briefly outlined in Section 1.2) will be elaborated upon in Chapter 5 in the cases of  $\mathcal{Fib}(1)$  and  $\mathcal{Fib}(2)$ .

### 3.5 Determining outer multiplicities of standard $\mathcal{Fib}$ -submodules

We will use the following shorthand notation for the outer multiplicity of a highest or lowest weight  $\lambda$  in  $\mathcal{Fib}(m)$ ,

$$M_m(\lambda) = M_{\mathcal{Fib}(m)}(\lambda).$$

For  $|m| > 2$ , Proposition 3.11 gives us that  $\mathcal{Fib}(m)_+$  and  $\mathcal{Fib}(m)_-$  are integrable  $\mathcal{Fib}$ -modules from category  $\mathcal{O}$  and  $\mathcal{O}^{op}$ , respectively, so

$$\mathcal{Fib}(m) = \left( \bigoplus_{\mu \in P'(\mathcal{Fib}(m))_+} M_m(\mu) V^\mu \right) \oplus \left( \bigoplus_{\lambda \in P'(\mathcal{Fib}(m))_-} M_m(\lambda) V^\lambda \right).$$

By the  $\psi$ -symmetry of  $\mathcal{Fib}(m)$ , it is enough to find the outer multiplicities of only positive weights, since for all  $\mu \in P'(\mathcal{Fib}(m))_+$ ,  $\mu = \psi(\lambda)$  for some  $\lambda \in P'(\mathcal{Fib}(m))_-$ , and moreover,  $M_m(\lambda) = M_m(\psi(\lambda)) = M_m(\mu)$ , hence

$$\mathcal{Fib}(m) = \bigoplus_{\lambda \in P'(\mathcal{Fib}(m))_-} M_m(\lambda) (V^\lambda \oplus V^{\psi(\lambda)}).$$

For  $0 < |m| \leq 2$ , Proposition 3.16 and the assumption that  $V^{\Lambda_m}$  is a direct summand of  $\mathcal{Fib}(m)$  gives us a similar decomposition,

$$\mathcal{Fib}(m) = V^{\Lambda_m} \oplus \bigoplus_{\lambda \in P'(\mathcal{Fib}(m))_-} M_m(\lambda) (V^\lambda \oplus V^{\psi(\lambda)}).$$

We wish to find the outer multiplicities  $M_m(\lambda)$  for all highest and lowest weights of  $\mathcal{Fib}$  in  $P(\mathcal{Fib}(m))$ . If we know the root multiplicities of  $\mathcal{F}$ , and if we know the inner multiplicities of 1) standard modules on each level (by Racah-Speiser), 2) the adjoint representation (by Kac-Peterson), and 3) the non-standard modules (by the recursion which will be detailed in Chapter 5), then the above decompositions hint at a recursive algorithm for finding the outer multiplicities of any level. This algorithm will be detailed in the next chapter, using the example of decomposing  $\mathcal{Fib}(0)$ .

## Chapter 4 Finding decomposition data for level 0

### 4.1 Trivial and adjoint representations of $\mathcal{F}ib$

We wish to express each  $\mathcal{F}ib(m)$  as a direct sum of irreducible  $\mathcal{F}ib$ -modules, starting with level 0. In [FF],  $\mathcal{A}ff$ -level 0 consisted only of  $\mathcal{A}ff$ . Similarly, we showed in Theorem 3.15 that the adjoint representation of  $\mathcal{F}ib$  is the unique irreducible non-standard module in  $\mathcal{F}ib(0)$ . We now find other  $\mathcal{F}ib$ -modules in  $\mathcal{F}ib(0)$ .

First we note that the Cartan of  $\mathcal{F}ib$  is two-dimensional while the Cartan of  $\mathcal{F}$  is three-dimensional. We now find a vector in  $\mathfrak{h}$  that commutes with all of  $\mathcal{F}ib$ , that is, the vector generates a one-dimensional trivial representation of  $\mathcal{F}ib$ .

**Theorem 4.1.** *The element  $C := -\frac{1}{2}(h_1 - 2h_3) \in \mathfrak{h}$  generates a trivial one-dimensional  $\mathcal{F}ib$ -module  $V^0 = \mathbb{C}C$  in  $\mathcal{F}ib(0)$ , that is,  $[C, \mathcal{F}ib] = 0$ , and  $\mathfrak{h} = \mathfrak{h}_{\mathcal{F}ib} \oplus V^0$ . Moreover,*

$$[C, X] = mX$$

for all  $X \in \mathcal{F}ib(m)$ , so the grading of  $\mathcal{F}$  by  $\mathcal{F}ib$  level,

$$\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \mathcal{F}ib(m),$$

is the eigenspace decomposition of  $\mathcal{F}$  with respect to  $ad_C$ , and  $[C, \mathcal{F}ib(0)] = 0$ .

*Proof.* Let  $C$  be a generator of a trivial  $\mathcal{F}ib$ -module. Proposition 3.8 showed that  $C$  must be a Cartan element, and moreover,  $\mathbb{Z}(\alpha_1 - 2\alpha_3)$  are the only solutions to  $\alpha(H_i) = 0$  for  $i = 1, 2$ . By Remark 1.20, since the Cartan matrices  $A$  and  $B(3)$  are both symmetric, we have for  $1 \leq i, j \leq 3$ ,  $\alpha_i(h_j) = \alpha_j(h_i)$ , and for  $1 \leq k, l \leq 2$ ,  $\beta_k(H_l) = \beta_l(H_k)$ . Extending linearly, this gives us  $\alpha(H_i) = 0 \Leftrightarrow \beta_i(H_\alpha) = 0$ , where  $H_\alpha \in \mathbb{R}(h_1 - 2h_3)$ . We choose  $C = -\frac{1}{2}(h_1 - 2h_3)$ .

Now suppose  $\alpha = x\alpha_1 + y\alpha_2 + z\alpha_3 \in \Delta_m$ . Consider  $X \in \mathcal{F}_\alpha$ . Then we have

$$\begin{aligned} [C, X] &= (x\alpha_1(-\frac{1}{2}(h_1 - 2h_3)) + y\alpha_2(-\frac{1}{2}(h_1 - 2h_3)) + z\alpha_3(-\frac{1}{2}(h_1 - 2h_3)))X \\ &= -\frac{1}{2}(x(2) + y(-2 - 2(-1)) + z(-2(2)))X = -(x - 2z)X \end{aligned}$$

Using the conversions  $x = b - a - c$ ,  $z = -c$ , and  $m = a - b - c$  (since  $\alpha \in \Delta_m$ ), it follows that  $[C, X] = -(b - a + c)X = mX$ .  $\square$

## 4.2 Highest and lowest-weight modules of $\mathcal{Fib}$ in level 0

We may now “complete” Proposition 3.16 to include the case  $m = 0$ , and show that  $Y(0) = \mathcal{Fib}(0)/(V^0 \oplus \mathcal{Fib})$  is a direct sum of highest- and lowest-weight  $\mathcal{Fib}$ -modules.

**Proposition 4.2** (cf. Proposition 3.16). *Let  $Y(0) = \mathcal{Fib}(0)/(V^0 \oplus \mathcal{Fib})$ . Then*

- i)  $P(Y(0))$  consists only of weights  $\mu$  where  $\|\mu\|^2 < 0$ , and
- ii)  $Y(0)$  is a sum of integrable highest-weight  $\mathcal{Fib}$ -modules in category  $\mathcal{O}$  (which are therefore completely reducible) and integrable lowest-weight  $\mathcal{Fib}$ -modules in category  $\mathcal{O}^{op}$  (which are also completely reducible), hence  $Y(0)$  is completely reducible.

*Proof.* Since  $V^0 \cap \mathcal{Fib} = \{0\}$ , and  $\dim(V^0) + \dim(\mathfrak{h}_{\mathcal{Fib}}) = 3 = \dim(\mathcal{F}_0)$ , we have that  $0 \notin P(Y(0))$ . The rest of the proof is identical to the proof of Proposition 3.16, only setting  $m = 0$ .  $\square$

**Proposition 4.3.**  *$\mathcal{Fib}$  is a direct summand of  $\mathcal{Fib}(0)$ .*

*Proof.* If  $\mathcal{Fib}$  is not a direct summand of  $\mathcal{Fib}(0)$ , then there necessarily exists a non-zero vector  $v \in \mathcal{Fib}(0)$  and  $\mathcal{Fib}$ -module  $\mathcal{Fib} \neq M \subset \mathcal{Fib}(0)$  such that  $v \in \mathcal{Fib} \cap M$ . Proposition 4.2 shows that  $M$  is either the trivial module or a direct sum of standard modules. Since  $v \in \mathcal{Fib}$  and  $\mathcal{Fib}$  is irreducible,  $v$  is a generator of  $\mathcal{Fib}$ . If  $v \in V^0$  then  $v$  is a generator of  $V^0$ , but since  $\mathcal{Fib}$  is irreducible, we have  $\mathcal{Fib} = V^0$ , a contradiction. Likewise, if  $v \in M$  then it is a generator of some irreducible standard module  $V^\lambda \subset M$  where  $\|\lambda\|^2 < 0$ , hence  $\mathcal{Fib} = V^\lambda$ , which is also a contradiction. Hence,  $v$  does not exist, and  $\mathcal{Fib}$  is a direct summand of  $\mathcal{Fib}(0)$ .  $\square$

Thus we have proven the following theorem, which proves the case of  $m = 0$  for Conjecture 3.7:

**Theorem 4.4.**

$$\mathcal{Fib}(0) = V^0 \oplus \mathcal{Fib} \oplus Y(0)_- \oplus Y(0)_+,$$

where  $Y(0)_-$  and  $Y(0)_+$  are direct sums of highest-weight and lowest-weight modules, respectively.

Thus finding the decomposition for  $\mathcal{Fib}(0)$  is equivalent to finding the decomposition

$$Y(0) = \bigoplus_{\lambda \in P'(Y(0))_-} M_0(\lambda)(V^\lambda \oplus V^{\psi(\lambda)}),$$

where  $V^\lambda$  and  $V^{\psi(\lambda)}$  are irreducible lowest-weight and highest-weight modules, respectively, and

$$P'(Y(0))_- = \{\mu \in \Delta_+^{im} \mid M_0(\lambda) > 0\} = \Delta \cap P_{\mathcal{Fib}}^-,$$

the set of negative dominant integral roots of  $\mathcal{Fib}$ . Thus for each  $\lambda \in P'(Y(0))_-$ , we must find the weight diagram for  $V^\lambda$ , its inner multiplicities, and outer multiplicity  $M_0(\lambda)$ . Section 3.4 details how to find weight diagram and inner multiplicities for  $V^\lambda$ . We now focus on finding  $M_0(\lambda)$ .

First note that  $\mu \in P'(Y(0))_-$  if and only if

$$Mult_{Y(0)_-}(\mu) = Mult_0(\mu) - Mult_{\mathcal{Fib}}(\mu) - \delta_{0,\mu} > 0,$$

and in general,

$$Mult_0(\mu) \geq Mult_{\mathcal{Fib}}(\mu),$$

with equality only occurring when  $\mu \in \Delta^{re}$ . This inequality becomes evident when one compares Figures 4.1a and 4.1b, which show the multiplicities of the positive roots of  $\mathcal{Fib}(0)$  up to height 12, and their multiplicities in  $\mathcal{Fib}$ , respectively [K2].

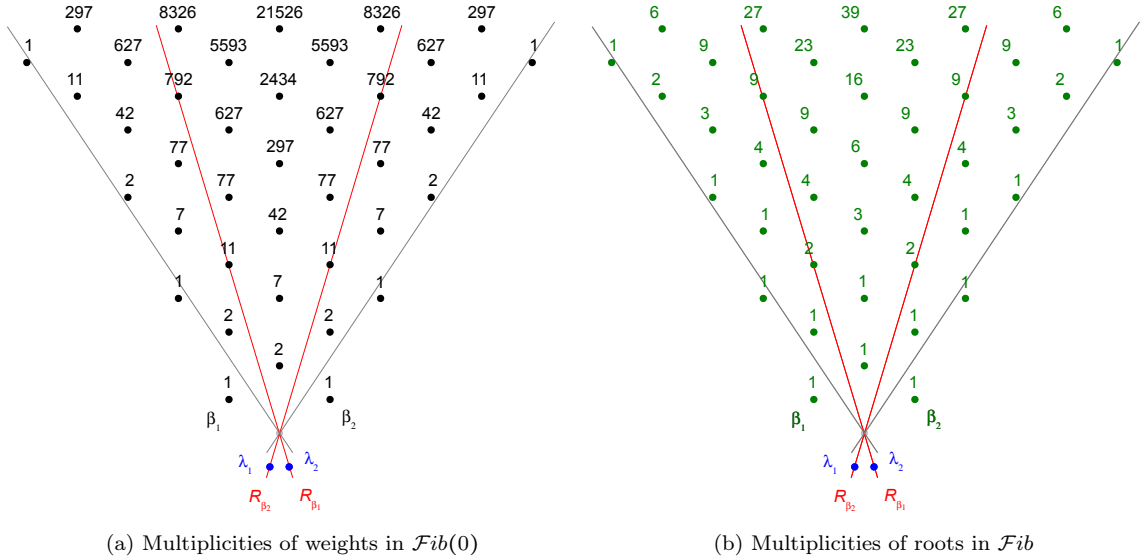


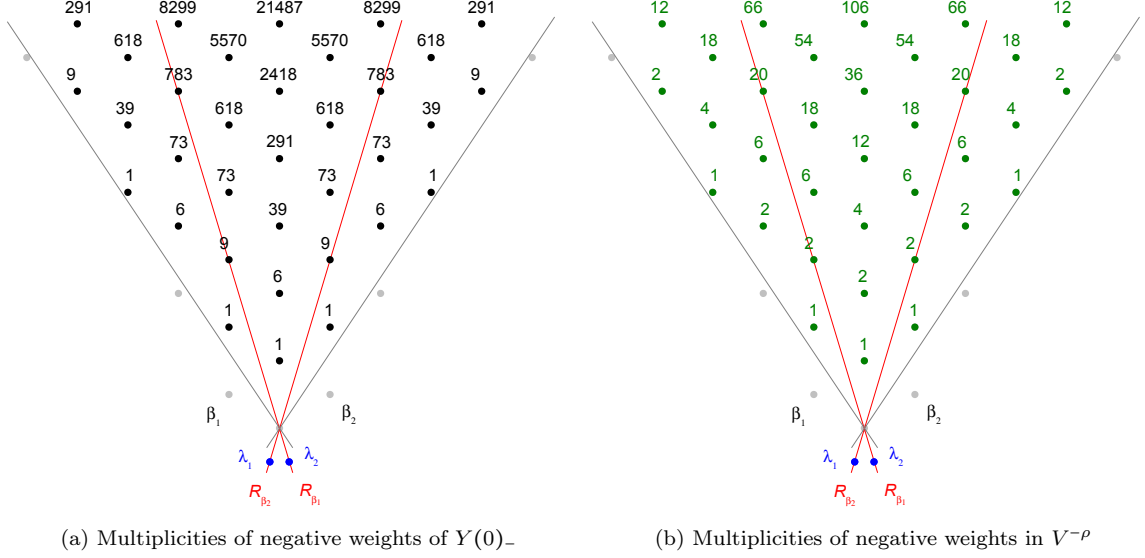
Figure 4.1: A comparison of the  $\mathcal{F}$ - and  $\mathcal{Fib}$ -multiplicities of positive roots on level 0.

Starting with the weight diagram  $P(\mathcal{Fib}(0)) = \Delta$ , one may construct the weight diagram  $P(Y(0))_-$  by deleting any weight  $\mu$  for which  $Mult_{Y(0)_-}(\mu) = 0$  (i.e., the projections of real roots, and 0). Then using the formula above one may label the remaining weights with their corresponding multiplicities in  $Y(0)_-$ . The result is shown in Figure 4.2a.

We now define a total order “ $>$ ” on  $P(Y(0))_-$  based on the partial order by height “ $>$ ” in the following way. For  $\lambda = n_1\beta_1 + n_2\beta_2$ ,  $\mu = m_1\beta_1 + m_2\beta_2 \in P(Y(0))_-$  (so  $n_i, m_i \geq 0$  for  $i = 1, 2$ ), define  $\lambda > \mu$  to be true if and only if either

- $\lambda > \mu$ , or
- $ht(\lambda - \mu) = 0$  and  $n_1 > m_1$ .




 Figure 4.2: A comparison of the multiplicities of negative weights of  $Y(0)_-$  and  $P(V^{-\rho})$ .

It is then clear from Figure 4.2a that

$$-\rho = -\rho_{\mathcal{Fib}} = -(\lambda_1 + \lambda_2) = \frac{1}{5} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \beta_1 + \beta_2$$

is the minimal height in  $P(Y(0))_-$  with respect to  $>$ , and  $M_0(-\rho) = Mult_{Y(0)_-}(-\rho) = 1$ .

It would appear then that  $\mathcal{Fib}_{\pm\rho}$  are the first examples of root spaces which contain extremal vectors for  $\mathcal{Fib}$  in  $\mathcal{Fib}(0)$ . To verify this claim, we seek a basis for  $Low_{\mathcal{Fib}(0)}(-\rho)$  that generates a  $\mathcal{Fib}$ -module  $V^{-\rho}$ . A basis for  $\mathcal{F}_{-\rho}$  is  $\{[E_2, 2E_1] = e_{21123}, e_{12123}\}$ , where the first basis element is in  $\mathcal{Fib}$ .

**Theorem 4.5.** *The set  $\{v_{-\rho} = -3e_{12123} + e_{21123}\}$  is a basis for  $Low_{\mathcal{Fib}(0)}(-\rho)$ , and generates  $V^{-\rho} \subset \mathcal{Fib}(0)$ , with outer multiplicity  $M_0(-\rho) = 1$ .*

*Proof.* We find  $v = a e_{12123} + b e_{21123} \neq 0$  such that  $F_1 \cdot v = 0 = F_2 \cdot v$ . We have

$$\begin{aligned} F_2 \cdot v &= [f_2, a e_{12123} + b e_{21123}] = a[f_2, e_{12123}] + b[f_2, e_{21123}] \\ &= a(e_{1213} + e_{1123}) + b(e_{2113} + 3e_{1123}) = a(e_{1123}) + b(3e_{1123}) \\ &= (a + 3b)e_{1123}, \end{aligned}$$

so  $F_2 \cdot v = 0$  if and only if  $a + 3b = 0$ . Now,

$$\begin{aligned} F_1 \cdot v &= \left[-\frac{1}{2}f_{1123}, a e_{12123}\right] + [F_1, b e_{21123}] = -\frac{a}{2}[f_{1123}, e_{12123}] + b[F_1, [E_2, 2E_1]] \\ &= -\frac{a}{2}[f_{1123}, e_{12123}] + 2b([E_1, [E_2, F_1]] + [E_2, -H_1]) \\ &= -\frac{a}{2}([e_{2123}, [e_1, f_{1123}]] + [e_1, [f_{1123}, e_{2123}]]) - 6bE_2 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{a}{2}[e_{2123}, 2f_{123}] - 6bE_2 = -a\left([f_{23}, [f_1, e_{2123}]] + [f_1, [e_{2123}, f_{23}]]\right) - 6bE_2 \\
 &= -a\left[f_1, [f_3, [f_2, e_{2123}]] + [f_2, [e_{2123}, f_3]]\right] - 6bE_2 \\
 &= -a\left[f_1, [f_3, e_{123}] + [f_2, e_{212}]]\right] - 6bE_2 = -a([f_1, -e_{12} - 2e_{21}]) - 6bE_2. \\
 &= -2(a + 3b)E_2,
 \end{aligned}$$

and once again,  $F_1 \cdot v = 0$  if and only if  $a + 3b = 0$ . Since the basis contains only one vector, we have  $M_0(-\rho) = 1$ .  $\square$

Thus we have begun to construct the decomposition of  $\mathcal{Fib}(0)$ ,

$$\mathcal{Fib}(0) = V^0 \oplus \mathcal{Fib} \oplus (V^{-\rho} \oplus V^{\rho}) \oplus \bigoplus_{\substack{\mu \in P'(Y(0))_- \\ \mu > -\rho}} M_0(\mu)(V^{\mu} \oplus V^{\psi(\mu)}).$$

Let

$$Y^1(0) = Y(0) / V^{-\rho} = \mathcal{Fib}(0) / (V^0 \oplus \mathcal{Fib} \oplus V^{-\rho} \oplus V^{\rho}),$$

so that  $P(Y^1(0))_- = P'(Y(0))_- \setminus \{-\rho\}$ . Then for all  $\mu \in P'(Y^1(0))_-$ ,

$$Mult_{Y^1(0)_-}(\mu) = Mult_{Y(0)_-}(\mu) - Mult_{-\rho}(\mu).$$

As before, the comparison in Figure 4.2 indicates that for all  $\mu \in P(Y^1(0))_-$ ,

$$Mult_{Y(0)_-}(\mu) \geq Mult_{-\rho}(\mu).$$

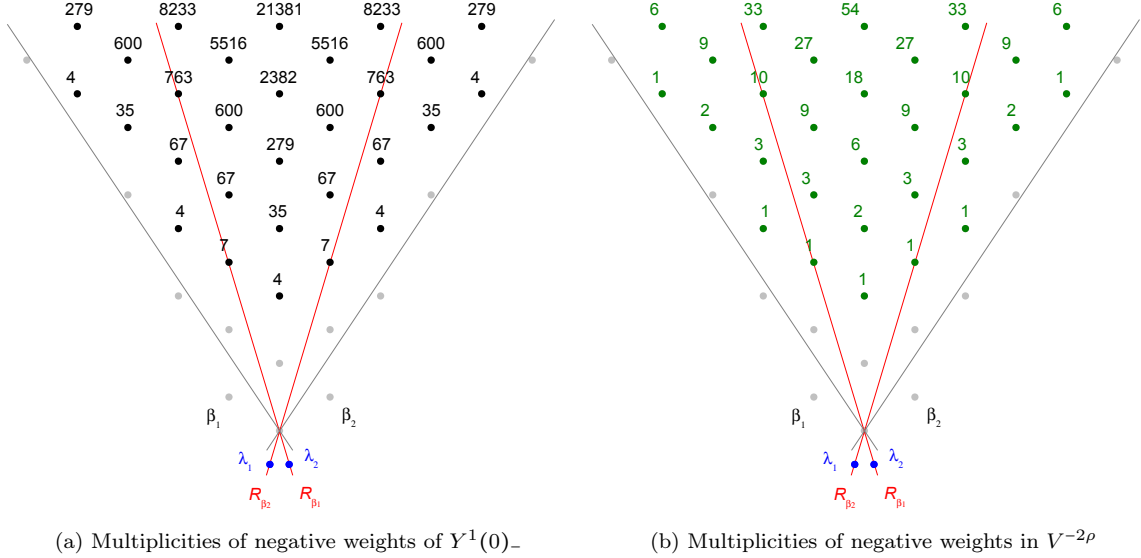


Figure 4.3: A comparison of the multiplicities of negative weights of  $Y^1(0)_-$  and  $P(V^{-2\rho})$ .

The weight diagram for  $Y^1(0)_-$  (Figure 4.3a) was found by deleting any  $\mu \in P(Y(0))_-$  for which equality holds (i.e., all  $W_{\mathcal{Fib}}$ -conjugates of  $-\rho$ ). The next weight in  $P'(Y(0))_-$  with respect to  $>$  is  $-2\rho$ . Figure 4.3a shows that  $M_0(-2\rho) = Mult_{Y^1(0)_-}(-2\rho) = 4$ , so

$$\mathcal{Fib}(0) = V^0 \oplus \mathcal{Fib} \oplus (V^{-\rho} \oplus V^{\rho}) \oplus 4(V^{-2\rho} \oplus V^{2\rho}) \oplus \bigoplus_{\mu \in P'(Y^2(0))_-} M_0(\mu)(V^{\mu} \oplus V^{\psi(\mu)}),$$

where

$$Y^2(0) = Y^1(0) / 4(V^{-2\rho} \oplus V^{2\rho}) = \mathcal{Fib}(0) / (V^0 \oplus \mathcal{Fib} \oplus V^{-\rho} \oplus V^{\rho} \oplus 4V^{-2\rho} \oplus 4V^{2\rho}),$$

so that  $P(Y^2(0))_- = P'(Y^1(0)_-) \setminus \{-2\rho\}$ . Then for all  $\mu \in P'(Y^2(0))_-$ ,

$$Mult_{Y^2(0)_-}(\mu) = Mult_{Y^1(0)_-}(\mu) - M_0(-2\rho)Mult_{-2\rho}(\mu).$$

Figure 4.3b shows the labeled weight diagram for  $P(V^{-2\rho})$ .

In general, for the  $n^{\text{th}}$  weight  $\mu_n$  in the total ordering of  $P'(Y(0))_-$ , if we define

$$Y^n(0) = Y^{n-1}(0) / M_0(\mu_n)(V^{\mu_n} \oplus V^{\psi(\mu_n)}) = \mathcal{Fib}(0) / (V^0 \oplus \mathcal{Fib} \oplus \bigoplus_{i=1}^n M_0(\mu_i)(V^{\mu_i} \oplus V^{\psi(\mu_i)})),$$

then for  $\mu \in P(Y(0))_-$ ,

$$Mult_{Y^n(0)_-}(\mu) = Mult_0(\mu) - \delta_{0,\mu} - Mult_{\mathcal{Fib}}(\mu) - \sum_{i=1}^n M_0(\mu_i)Mult_{\mu_i}(\mu),$$

and we have the decomposition of level 0,

$$\mathcal{Fib}(0) = V^0 \oplus \mathcal{Fib} \oplus \bigoplus_{\mu \in P'(Y(0))_-} M_0(\mu)(V^{\mu} \oplus V^{\psi(\mu)}).$$

$\lambda \in P'(Y(0))_-$	$M_0(\lambda)$
$-\rho$	1
$5\lambda_1$ and $5\lambda_2$	3
$-2\rho$	4
$-3\rho$	21
$6\lambda_1 + \lambda_2$ and $\lambda_1 + 6\lambda_2$	28
$10\lambda_1$ and $10\lambda_2$	135
$-4\rho$	145
$7\lambda_1 + 2\lambda_2$ and $2\lambda_1 + 7\lambda_2$	254
$-5\rho$	1182
$11\lambda_1 + \lambda_2$ and $\lambda_1 + 11\lambda_2$	2184
$8\lambda_1 + 3\lambda_2$ and $3\lambda_1 + 8\lambda_2$	2375
$-6\rho$	10349

Table 4.1: The sequence of outer multiplicities of irreducible LW  $\mathcal{Fib}$ -modules in level 0, ordered by increasing  $M_0(\lambda)$

Continuing in this fashion will lead to more data on the location of lowest-weight vectors for  $\mathcal{Fib}$ , and the dimensions of the corresponding subspaces in  $\mathcal{Fib}(0)$ . However, as the last

theorem suggests, calculating actual bases of extremal vectors using linear algebra will get increasingly difficult as the dimensions increase, and in the end this method will not provide any insight into how these extremal vectors arise, nor does it seem to hint at a simpler method of generating them. Ignoring for now the determination of bases of extremal vectors, and focusing instead on locating lowest weights and quotienting the appropriate number of copies of the corresponding modules in the way described above, our investigation has produced data on outer multiplicities of highest and lowest weights on level 0 shown in Table 4.1. The dimensions of the extremal weight spaces themselves appear to not follow any recognizable pattern.

## Chapter 5 Finding decomposition data for levels $\pm 1, \pm 2$

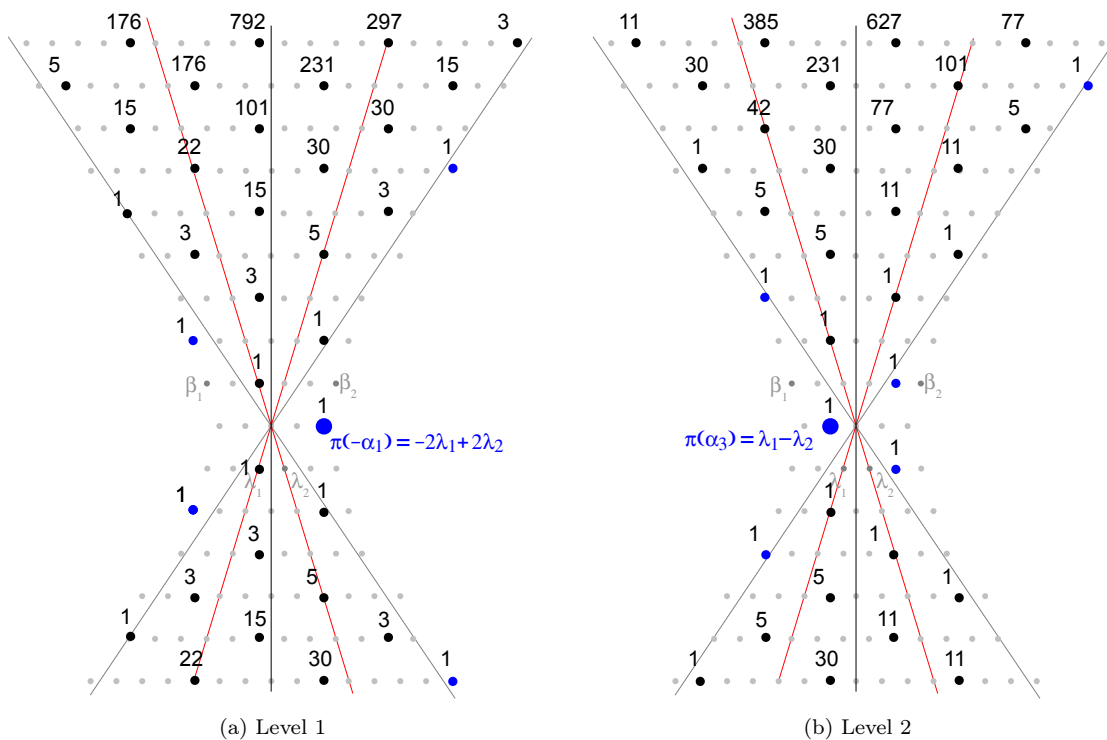


Figure 5.1: Partial weight diagrams for  $\mathcal{F}ib(m), m = 1, 2$ . The multiplicities shown are  $Mult_m(\mu) = \dim_{\mathcal{F}}(\mathcal{F}_{\mu})$  for weights  $\mu$  such that  $-9 \leq wt(\mu) \leq 6$ , and are upper bounds for the inner multiplicities of the irreducible non-standard module  $V^{\Lambda_m}$ .

### 5.1 Inner multiplicities of the non-standard $\mathcal{Fib}$ -modules on Levels 1, 2

Figure 5.1 shows the weight diagrams for levels 1 and 2, for weights  $\mu$  such that  $-9 \leq wt(\mu) \leq 6$  (cf. also Figure 3.1). Both diagrams include weight multiplicities in  $\mathcal{F}$ , computed by Kac using the Kac-Peterson recursion (see Appendix B) and listed in Table  $H_3$  in Ch. 11 of [K2]. Recall that since  $\nu(\mathcal{F}ib(m)) = \mathcal{F}ib(-m)$ , we only consider  $m > 0$ . We know from Proposition 3.15 that there is one non-standard quotient module  $V(m)/U(m)$  on levels  $m = \pm 1, \pm 2$ . Assuming  $U(m) = \{0\}$  for each of these levels, we have

$$V^{\Lambda_1} = \mathcal{U}(\mathcal{Fib}) \cdot f_1, \quad V^{\Lambda_{-1}} = \mathcal{U}(\mathcal{Fib}) \cdot e_1, \quad V^{\Lambda_2} = \mathcal{U}(\mathcal{Fib}) \cdot e_3, \quad \text{and} \quad V^{\Lambda_{-2}} = \mathcal{U}(\mathcal{Fib}) \cdot f_3,$$

and by Proposition 3.16 we have that

$$Y(\pm 1) = \mathcal{Fib}(\pm 1) / V^{\Lambda_{\pm 1}} \quad \text{and} \quad Y(\pm 2) = \mathcal{Fib}(\pm 1) / V^{\Lambda_{\pm 2}}$$

are completely reducible. Using the same method prescribed in Section 4.2, we can locate extremal weights for  $\mathcal{Fib}$  on levels  $\pm 1$  and  $\pm 2$ , and determine their outer multiplicities. However, this procedure assumes knowledge of the inner multiplicities for  $V^{\Lambda_{\pm m}}$ ,  $|m| = 1, 2$ , which we do not yet have. The Racah-Spesier recursion can only be applied to standard modules, and although the Kac-Peterson recursion works for the adjoint representation of any KM algebra (which is non-standard) there is no reason to believe it can apply to any nonstandard module.

On page 12 in Section 1.2, we gave a brief outline for how one may determine weight multiplicities of an irreducible highest-weight  $\mathcal{L}$ -module  $V^\lambda$  by recursively computing bases for its weight spaces. This method, in principal, will work for any irreducible module, including non-standard ones. We introduce the following definitions which will be used in describing the method.

**Definition 5.1.** *Let  $m = 1, 2$ . If  $\mu = \Lambda_m + n_1\beta_2 + n_2\beta_2 \in P(V^{\Lambda_m})$ , then  $\mu$  is uniquely determined by the coefficients of  $\beta_1, \beta_2$ , so we may write  $\mu = (n_1, n_2)_m$ . We suppress the subscript if there is no ambiguity as to the level in question.*

As in Chapter 4 we utilize  $\psi$ -symmetry and consider only negative weights  $P(V_-^{\Lambda_m})$ , which has a total order similar to that of  $P(Y(0))_-$ .

**Definition 5.2.** *For  $\lambda = (n_1, n_2)$  and  $\mu = (m_1, m_2)$ , define  $\lambda > \mu$ , to be true if and only if either*

- $ht(\lambda - \mu) > 0$ , or
- $ht(\lambda - \mu) = 0$  and  $n_1 > m_1$ .

**Definition 5.3.** *Let  $V^\lambda = \mathcal{U}(\mathcal{Fib})v^\lambda$  be a non-standard  $\mathcal{Fib}$ -module with generating vector  $v^\lambda \in V_\lambda^\lambda$ . The notation  $[i_1 i_2 \dots i_n]_\lambda$  stands for  $E_{i_1} E_{i_2} \dots E_{i_n} v^\lambda = E_{i_1} \cdot (E_{i_2} \cdot (\dots (E_{i_n} \cdot v^\lambda) \dots))$ , and  $v^\lambda$  is denoted by  $[]_\lambda$ . (The  $\lambda$  in the subscript will be suppressed unless more than one projected level is being discussed.)*

We now elaborate on the procedure for determining inner multiplicities of an irreducible module  $V^{\Lambda_m}$ , using the example for  $m = 1$ . By Proposition 3.15 we know that  $P(V^{\Lambda_1}) = P(\mathcal{Fib}(1))$ . Figure 5.2a shows the partial weight diagram, labeled with computed inner multiplicities. The lowest weight in the total order  $<$  is  $\Lambda_1 = \pi(-\alpha_1)$ . Since  $-\alpha_1$  is a real root,  $Mult_{\Lambda_1}(\Lambda_1) = 1$ , so  $V_{\Lambda_1}^{\Lambda_1}$  has basis consisting of the single vector  $v_{\Lambda_1} = f_1$ .

In general, let  $\Lambda_1 < \mu = (n_1, n_2) \in P(V^{\Lambda_1})$ . Assume we have previously determined a basis  $\mathcal{B}_{\mu-\beta_1}$  for  $V_{\mu-\beta_1}^{\Lambda_1}$  and a basis  $\mathcal{B}_{\mu-\beta_2}$  for  $V_{\mu-\beta_2}^{\Lambda_1}$ . If for some  $i = 1$  or  $2$ ,  $\mu - \beta_i \notin P(V^{\Lambda_1})$ , then  $\mathcal{B}_{\mu-\beta_i} = \emptyset$ . Obtain a spanning set for  $V_{\mu}^{\Lambda_1}$ ,

$$\mathcal{S}_{\mu} = \mathcal{S}_{(n_1, n_2)} = (E_1 \cdot \mathcal{B}_{\mu-\beta_1}) \cup (E_2 \cdot \mathcal{B}_{\mu-\beta_2}) = \{v_1, \dots, v_k\}$$

of size  $k = |\mathcal{B}_{\mu-\beta_1}| + |\mathcal{B}_{\mu-\beta_2}|$ . Determine if there are any linear dependence relations on the vectors in  $\mathcal{S}_{\mu}$  by solving the homogeneous system of linear equations

$$F_i \cdot \sum_{j=1}^k c_j v_j = \sum_{j=1}^k c_j (F_i \cdot v_j) = 0 \quad \text{for } i = 1, 2. \quad (5.1)$$

Since  $V_{\mu}^{\Lambda_1}$  cannot contain any lowest weight vectors, any nontrivial solutions will yield dependence relations on the vectors in  $\mathcal{S}_{\mu}$ . Then choose vectors to delete from the spanning set to obtain a basis  $\mathcal{B}_{\mu}$  for  $V_{\mu}^{\Lambda}$ . Finally, we have  $Mult_{\lambda}(\mu) = |\mathcal{B}(\mu)|$ .

**Remark 5.4.** If  $v_j \in \mathcal{B}_{\mu-\beta_j}$  for  $j = 1, 2$ , then for  $i = 1, 2$  the Jacobi identity gives us

$$\begin{aligned} F_i \cdot (E_j \cdot v_j) &= E_j \cdot (F_i \cdot v_j) - [E_j, F_i] \cdot v_j = E_j \cdot (F_i \cdot v_j) - \delta_{ij} H_j \cdot v_j \\ &= E_j \cdot (F_i \cdot v_j) - \delta_{ij} (\mu - \beta_j)(H_j) v_j. \end{aligned}$$

Thus we observe that for  $i, j = 1, 2$ ,  $F_i \cdot (E_j \cdot v_j)$  is determined by  $(\mu - \beta_j)(H_j) = \mu(\beta_j) - 2$  and our recursive knowledge of  $F_i \cdot v_j$  for  $v_j \in \mathcal{B}_{\mu-\beta_j}$ .

Remark 5.4 reveals a useful recursive approach to solving the system ((5.1)), which involve complicated multibracket computations. In Table C1 of Appendix C, we present the following data:

- The first column shows  $\mu = (n_1, n_2) \in P(\Lambda_1)$ . The table is sorted by this first column, ordered by  $<$ .
- The second column lists, for each  $\mu \in P(\Lambda_1)$ , an *ordered* spanning set

$$\mathcal{S}_{\mu} = (E_2 \cdot u_1, \dots, E_2 \cdot u_m, E_1 \cdot v_1, \dots, E_1 \cdot v_n) \quad (5.2)$$

where  $\mathcal{B}_{\mu-\beta_2} = (u_1, \dots, u_m)$  and  $\mathcal{B}_{\mu-\beta_1} = (v_1, \dots, v_n)$  are ordered bases. (There is some justification for this ordering of the set  $\mathcal{S}_{\mu}$ , which will be discussed shortly.) The vectors are written using the shorthand notation from Definition 5.3.

- The third column contains a ‘\*’ if there is a dependence relation found on  $\mathcal{S}_{\mu}$ . The vectors deleted from  $\mathcal{S}_{\mu}$  to form the basis  $\mathcal{B}_{\mu}$  are then shown in red, and their coordinates with respect to  $\mathcal{B}_{\mu}$  are given.

- The fourth and fifth columns show, for each  $u \in \mathcal{S}_\mu$  in the second column, the result of the computation  $F_1 \cdot u$  (in column 4) and  $F_2 \cdot u$  (in column 5), written as coordinate vectors  $(c_1, c_2, \dots, c_k)_{\mu-\beta_i}$  with respect to the basis  $\mathcal{B}_{\mu-\beta_i}$  (so  $k = n$  if  $i = 1$  and  $k = m$  if  $i = 2$ ).
- For  $u \in \mathcal{S}_\mu$ , the sixth column shows  $\mu(H_1)$ , and the seventh column shows  $\mu(H_2)$ .

We now demonstrate the recursive algorithm for determining bases for weight spaces in  $V^{\Lambda_1}$  by showing how the first several rows of Table C1 were computed. First, we have

$$\mathcal{S}_{(0,0)} = \{v_{\Lambda_1} = []\} = \mathcal{B}_{(0,0)},$$

and  $Mult_{\Lambda_1}(\Lambda_1) = 1$ . Furthermore, the weight diagram shows that  $F_1 \cdot [] = 0$ , and

$$\Lambda_1(H_1) = (-2\lambda_1 + 2\lambda_2)(H_1) = -2, \quad \text{and} \quad \Lambda_1(H_2) = (-2\lambda_1 + 2\lambda_2)(H_2) = 2.$$

( $F_2 \cdot []$  will not be needed for the recursion, as we will see.) Since  $V_{(0,0)}^{\Lambda_1}$  is a one-dimensional space there are no dependence relations, and we have completed row 1 of Table C1. For row 2, we have

$$\mathcal{S}_{(1,0)} = \{E_1 \cdot v_{\Lambda_1}\} = \{[1]\} = \mathcal{B}_{(1,0)}$$

and  $Mult_{\Lambda_1}((1,0)) = 1$ , since  $E_2 \cdot v_{\Lambda_1} = 0$  according to the weight diagram. Then using Remark 5.4 and the data in row 1 of Table C1, we have

$$F_1 E_1 \cdot v_{\Lambda_1} = (E_1 F_1 - H_1) \cdot v_{\Lambda_1} = -H_1 \cdot v_{\Lambda_1} = -\Lambda_1(H_1) v_{\Lambda_1} = 2v_{\Lambda_1}.$$

We write this vector with respect to the basis  $\mathcal{B}_{(0,0)} = (v_{\Lambda_1})$ , entering it into the table as  $(2)_{(0,0)}$ . Note that the weight diagram also reveals that  $F_2 E_2 \cdot v_{\Lambda_1} = 0$ , confirming there was no need to compute  $F_2 \cdot []$ . We complete row 2 of Table C1 by computing

$$(\beta_1 + \Lambda_1)(H_1) = \beta_1(H_1) + \Lambda_1(H_1) = 0 \quad \text{and} \quad (\beta_1 + \Lambda_1)(H_2) = \beta_1(H_2) + \Lambda_1(H_2) = -1.$$

We then have

$$\mathcal{S}_{(2,0)} = \{E_1 E_1 \cdot v_{\Lambda_1}\} = \{[11]\} = \mathcal{B}_{(2,0)} \quad \text{and} \quad \mathcal{S}_{(1,1)} = \{E_2 E_1 \cdot v_{\Lambda_1}\} = \{[21]\} = \mathcal{B}_{(1,1)},$$

so  $Mult_{\Lambda_1}((2,0)) = Mult_{\Lambda_1}((1,1)) = 1$ . For  $\mathcal{B}_{(2,0)}$  we have

$$F_1[11] = F_1 E_1[1] = (E_1 F_1 - H_1)[1] = E_1(2[]) - (0)[1] = 2[1] = (2)_{(1,0)},$$

$$F_2[11] = 0 \quad (\text{immediately follows from the weight diagram}),$$

$$(2\beta_1 + \Lambda_1)(H_1) = 2\beta_1(H_1) + \Lambda_1(H_1) = 2, \quad \text{and}$$

$$(2\beta_1 + \Lambda_1)(H_2) = 2\beta_1(H_2) + \Lambda_1(H_2) = -4.$$



For  $\mathcal{B}_{(1,1)}$  we have

$$F_1[21] = 0 \text{ (immediately follows from the weight diagram),}$$

$$F_2[21] = F_2 E_2[1] = (E_2 F_2 - H_2)[1] = E_2(0) - (-1)[1] = [1] = (1)_{(1,0)},$$

$$\beta_2(H_1) + (\beta_1 + \Lambda_1)(H_1) = -3, \text{ and}$$

$$\beta_2(H_2) + (\beta_1 + \Lambda_1)(H_2) = 1.$$

Next we have

$$\mathcal{S}_{(2,1)} = (E_2 \cdot \mathcal{B}_{(2,0)}) \cup (E_1 \cdot \mathcal{B}_{(1,1)}) = \{[211], [121]\},$$

and

$$F_1[211] = E_2 F_1[11] = E_2(2[1]) = 2[21] = (2)_{(1,1)}$$

$$F_2[211] = (E_2 F_2 - H_2)[11] = E_2(0) - (-4)[11] = (4)_{(2,0)},$$

and

$$F_1[121] = (E_1 F_1 - H_1)[21] = E_1(0) - (-3)[21] = 3[21] = (3)_{(1,1)}$$

$$F_2[121] = (E_1 F_2)[21] = E_1([1]) = (1)_{(2,0)},$$

and  $(2\beta_1 + \beta_2)(H_1) = -1$ ,  $(2\beta_1 + \beta_2)(H_2) = -2$  for both vectors. Then, the solution to the system of equations

$$c_1 F_1[211] + c_2 F_1[121] = 0,$$

$$c_1 F_2[211] + c_2 F_2[121] = 0,$$

is exactly the null space of the matrix  $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ , which is trivial. No dependence relations exist, so  $\mathcal{B}_{(2,1)} = \mathcal{S}_{(2,1)}$  and  $Mult_{\Lambda_1}((2,1)) = 2$ . Observe that this matrix is the transpose of the matrix below formed from the cells in Table C1 corresponding to  $F_i \cdot \mathcal{S}_{(2,1)}$  for  $i = 1, 2$ :

$$A(\mu) = A(2,1) = \begin{bmatrix} (F_1[211])_{(1,1)} & (F_2[211])_{(2,0)} \\ (F_1[121])_{(1,1)} & (F_2[121])_{(2,0)} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}^T,$$

and linear dependence relations on  $V_{(2,1)}^{\Lambda_1}$  are defined by basis vectors in  $Null(A(2,1)^T)$ . In fact, this generalizes for all  $\mu = (n_1, n_2) \in P(V^{\Lambda_1})$ , and it is the reason we choose a consistent ordering for the spanning sets  $\mathcal{S}_\mu$ . In general, we have the block-form  $(m+n) \times (m+n)$  matrix (recall from ((5.2)) that  $m = \dim V_{\mu-\beta_2}^{\Lambda_1}$ ,  $n = \dim V_{\mu-\beta_1}^{\Lambda_1}$ ),

$$A_\mu = \begin{bmatrix} \left( F_1(E_2 \mathcal{B}_{\mu-\beta_2}) \right)_{\mu-\beta_1}^T & \left( F_2(E_2 \mathcal{B}_{\mu-\beta_2}) \right)_{\mu-\beta_2}^T \\ \left( F_1(E_1 \mathcal{B}_{\mu-\beta_1}) \right)_{\mu-\beta_1} & \left( F_2(E_1 \mathcal{B}_{\mu-\beta_1}) \right)_{\mu-\beta_2} \end{bmatrix}$$

$$= \begin{bmatrix} \left( (E_2 F_1) \mathcal{B}_{\mu-\beta_2} \right)_{\mu-\beta_1}^T & \left( (E_2 F_2 - H_2) \mathcal{B}_{\mu-\beta_2} \right)_{\mu-\beta_2}^T \\ \left( (E_1 F_1 - H_1) \mathcal{B}_{\mu-\beta_1} \right)_{\mu-\beta_1}^T & \left( (E_1 F_2) \mathcal{B}_{\mu-\beta_1} \right)_{\mu-\beta_2}^T \end{bmatrix},$$

where the rows of each block are coordinates of vectors  $F_i \cdot (E_j \cdot \mathcal{B}_{\mu-\beta_j})$  for  $i, j = 1, 2$  with respect to the basis  $\mathcal{B}_{\mu-\beta_i}$ . Then

$$A_\mu^T = \begin{bmatrix} \left( (E_2 F_1) \mathcal{B}_{\mu-\beta_2} \right)_{\mu-\beta_1} & \left( (E_1 F_1 - H_1) \mathcal{B}_{\mu-\beta_1} \right)_{\mu-\beta_1} \\ \left( (E_2 F_2 - H_2) \mathcal{B}_{\mu-\beta_2} \right)_{\mu-\beta_2} & \left( (E_1 F_2) \mathcal{B}_{\mu-\beta_1} \right)_{\mu-\beta_2} \end{bmatrix}.$$

The author has found additional interesting recursive patterns relating the blocks of  $A_\mu^T$  to the blocks of  $A_{\mu-\beta_i}^T$  for each  $i = 1, 2$ , that further reduced computation time. These recursions can then be used to automate the algorithm described above using Mathematica.

If a dependence relation is found on  $\mathcal{S}_\mu$ , then in column 2 of Table C1 we color-code in red all vectors deleted from  $\mathcal{S}_\mu$  in the formation of basis  $\mathcal{B}_\mu$ , and indicate the associated dependence relations on  $\mathcal{S}_\mu$  by writing these deleted vectors as coordinate vectors with respect to  $\mathcal{B}_\mu$ . The first example encountered in the table is for  $\mu = (4, 1)$ , where

$$\mathcal{S}_{(4,1)} = \{[11211], [11121]\}.$$

Since  $\mu - \beta_2 \notin P(V_+^{\Lambda_1})$ , we have  $(\mathcal{B}_{(4,0)})_{3,1} = \emptyset$ , so

$$A(4, 1)^T = \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix},$$

which gives us the linear dependence relation  $-\frac{3}{2}[11211] + [11121] = 0$ . We then let  $\mathcal{B}_{(4,1)} = \mathcal{S}_{(4,1)} - \{[11211]\}$ , and we write  $[11211] = \frac{2}{3}[11121]$  in the second column.

This algorithm was repeated for all  $\mu \in P(V^{\Lambda_1})$  shown in Figure 5.2a, though in principle it can be continued indefinitely to gain more inner multiplicity data. The algorithm was then applied to the non-standard module  $V^{\Lambda_2}$ , and the results are recorded in Table C2. The data in Tables C1 and C2 can then be used to determine inner multiplicities since  $Mult_{\Lambda_m}(\mu) = \dim_{V^{\Lambda_m}}(V_\mu^{\Lambda_m}) = |\mathcal{B}_\mu|$ . Figure 5.2 shows the resulting weight diagrams for non-standard modules on levels 1 and 2, labeled with inner multiplicities.

We observe that the weights in Figure 5.2 do not follow the Kac-Peterson recursion (cf. Appendix A), but instead follow a Racah-Speiser recursion (cf. Figure 3.3).

**Conjecture 5.5.** *For  $m = \pm 1, \pm 2$ , the weights of  $P(V_-^{\Lambda_m})$  follow the Racah-Speiser recursion*

$$Mult_\lambda(\mu) = - \sum_{1 \neq w \in W_{\mathcal{Fib}}} \det(w) Mult_\lambda(\mu + (w\rho - \rho)),$$

*and the weights of  $P(V_+^{\Lambda_m})$  follow the Racah-Speiser recursion*

$$Mult_\lambda(\mu) = - \sum_{1 \neq w \in W_{\mathcal{Fib}}} \det(w) Mult_\lambda(\mu - (w\rho - \rho)),$$

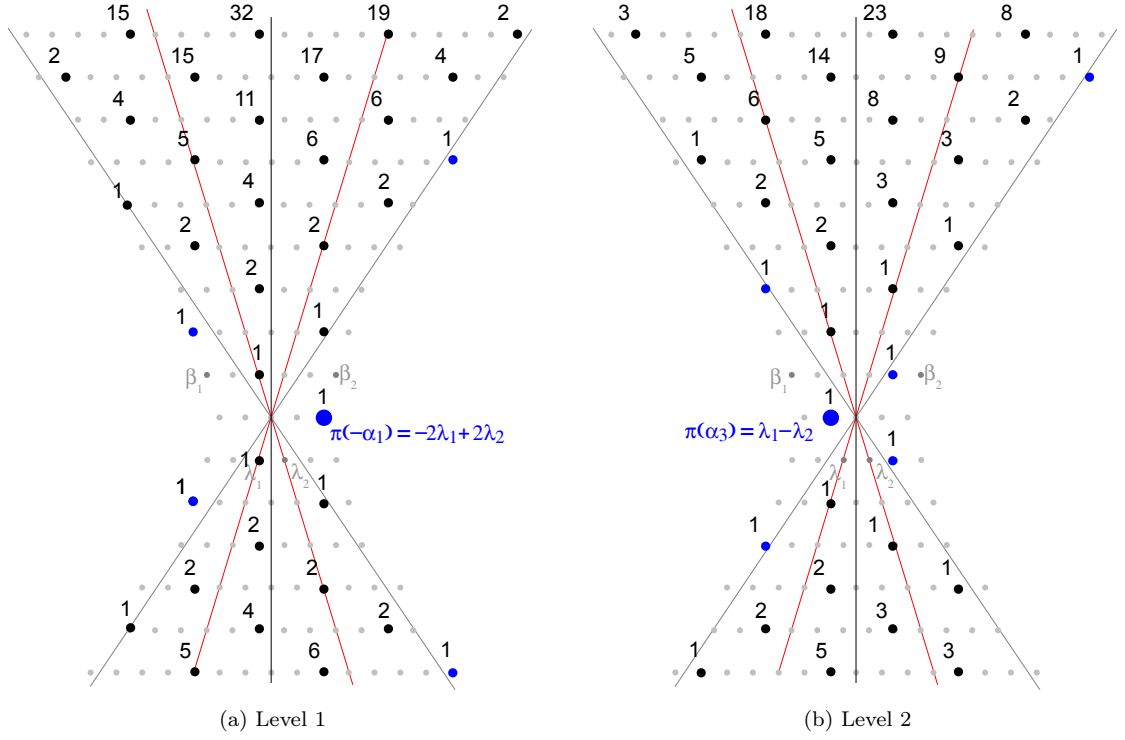


Figure 5.2: Partial weight diagrams for non-standard  $\mathcal{Fib}$ -modules  $V^{\Lambda_m}$  for  $m = 1$  and  $2$ . The inner multiplicities shown are  $Mult_{\Lambda_m}(\mu) = \dim_{V^{\Lambda_1}}(V_{\mu}^{\Lambda_1})$  for weights  $\mu$  such that  $-9 \leq wt(\mu) \leq 6$  and were calculated using the recursive algorithm presented in Section 5.1 (cf. Tables C1 and C2).

where  $\rho = \lambda_1 + \lambda_2$ .

If this conjecture is true, then the non-standard modules on levels  $\pm 1, \pm 2$  have more in common with highest- and lowest-weight modules than with the non-standard module (adjoint representation) on level 0.

## 5.2 Outer multiplicities of $\mathcal{Fib}(\pm 1)$ and $\mathcal{Fib}(\pm 2)$

We may now use the method described in Section 4.2 to locate extremal weights for  $\mathcal{Fib}$  on levels  $\pm 1, \pm 2$ , and determine their outer multiplicities. For example, Figure 5.3a shows a portion of the weight diagram of the (completely reducible) quotient module  $Y(1) = \mathcal{Fib}/V^{\Lambda_1}$ , with multiplicities  $Mult_{Y(1)}(\mu) = Mult_1(\mu) - Mult_{\Lambda_1}(\mu)$ . We observe from this diagram that  $(2, 1)$  is a lowest weight for  $\mathcal{Fib}$ , and using Racah-Speiser we find the labeled weight diagram for  $V^{(2,1)}$  shown in Figure 5.3b. These two diagrams then give us the labeled weight diagram for the quotient module

$$Y^1(1) = \mathcal{Fib}(1) / (V^{\Lambda_1} \oplus V^{(2,1)} \oplus V^{\psi(2,1)})$$

shown in Figure 5.3c. We then observe that  $M_1((2, 2)) = 2$ .

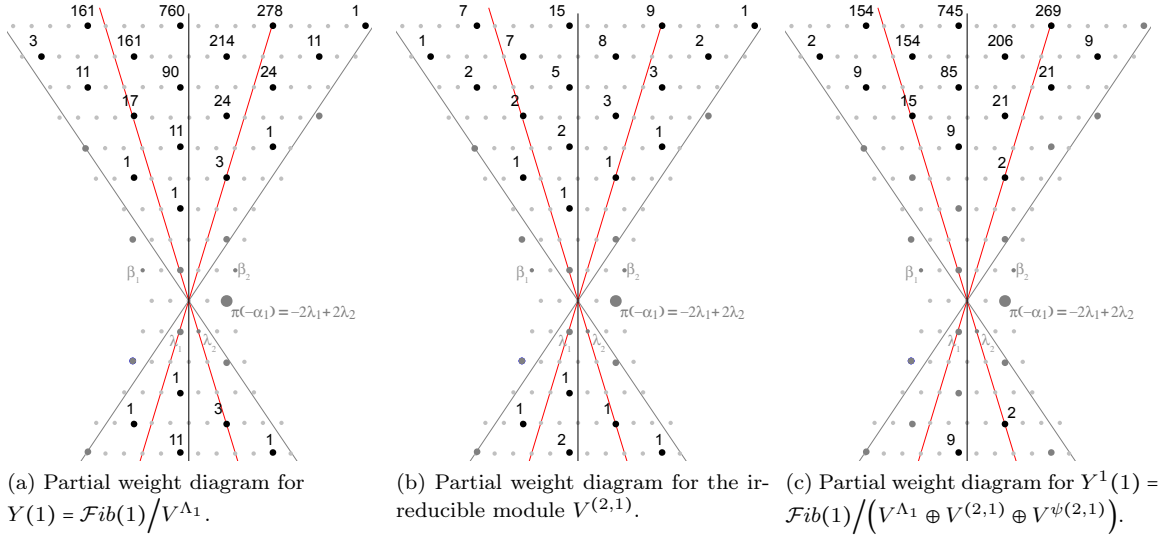


Figure 5.3: Partial weight diagrams for three  $\mathcal{Fib}$ -modules on level 1 labeled with their multiplicities. Notation  $(n_1, n_2)$  refers to weight  $\Lambda_1 + n_1\beta_1 + n_2\beta_2$ , since these diagrams are all in  $\mathcal{Fib}(1)$ .

Continuing in this way gives some data on outer multiplicities for level 1, presented in Table 5.1a. The table is ordered by increasing outer multiplicity  $M_1(\lambda)$ .

$\lambda = (n_1, n_2)_1$	$M_1(\lambda)$
(2, 1)	1
(2, 2)	2
(4, 2)	6
(3, 2)	7
(3, 3)	12
(4, 3)	49
(4, 5)	54
(5, 3)	67
(4, 4)	100
(5, 4)	385

(a) Outer multiplicity data for level 1.

$\lambda = (n_1, n_2)_2$	$M_2(\lambda)$
(2, 2)	3
(2, 3)	5
(3, 3)	14
(4, 3)	16
(3, 5)	20
(3, 4)	36
(4, 4)	107
(4, 5)	295

(b) Outer multiplicity data for level 2.

Table 5.1: The sequence of outer multiplicities of irreducible LW  $\mathcal{Fib}$ -modules in levels 1 and 2, where notation  $(n_1, n_2)_m$  is as defined in Definition 5.1.

Similarly, for level 2 we have Figure 5.4, which shows portions of the weight diagrams for the (completely reducible) quotient module  $Y(2) = \mathcal{Fib}/V^{\Lambda_2}$ , the irreducible lowest-weight module  $V^{(2,2)}$  (which has inner multiplicity 5), and the (completely reducible) quotient module

$$Y^1(2) = \mathcal{Fib}(2)/(V^{\Lambda_1} \oplus V^{(2,2)} \oplus V^{\psi(2,2)}).$$

We then observe that  $(2, 2)$  is a lowest weight for  $\mathcal{Fib}$  with outer multiplicity  $M_2((2, 2)) = 3$ . As before, we may continue the procedure indefinitely, obtaining more data on extremal

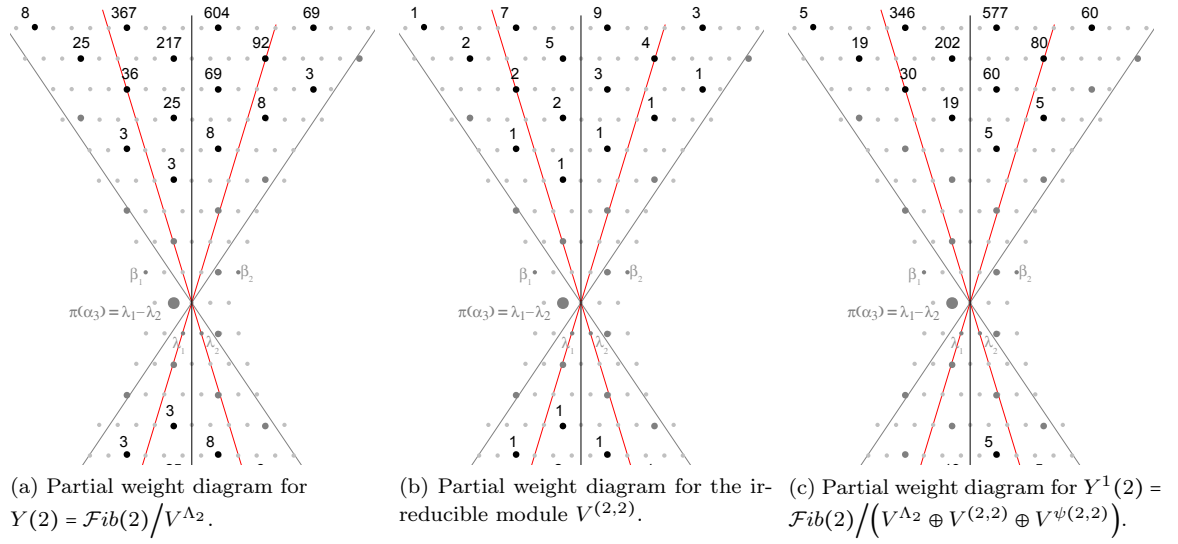


Figure 5.4: Partial weight diagrams for three  $\mathcal{Fib}$ -modules on level 2 labeled with their multiplicities. Notation  $(n_1, n_2)$  refers to weight  $\Lambda_2 + n_1\beta_1 + n_2\beta_2$ , since these diagrams are all in  $\mathcal{Fib}(2)$ .

vectors and their outer multiplicities. Table 5.1b presents the sequence of outer multiplicities for extremal weights on level 2, ordered by increasing outer multiplicity  $M_2(\lambda)$ .

## Chapter 6 The Vertex algebra approach

We observe that the algorithmic approach to finding extremal vectors for  $\mathcal{Fib}$  described in Chapters 4 and 5 is time-intensive, and the sequences of outer multiplicities produced do not appear to follow any recognizable pattern. We therefore seek an alternative approach using the theory of vertex algebras, which may give some insight into the decomposition of  $\mathcal{F}$  with respect to  $\mathcal{Fib}$ .

### 6.1 Definitions and the vertex algebra $V_{\mathcal{Fib}}$

In [B] Borchers gave a prescription for constructing a vertex algebra from any lattice, including indefinite lattices. In this section, we apply Borchers' method to the indefinite root lattice  $Q_{\mathcal{Fib}} = \mathbb{Z}\beta_1 \oplus \mathbb{Z}\beta_2$ , and in Section 6.2 we apply it to  $Q_{\mathcal{F}} = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3$ .

Consider the Fock space  $S(\hat{\mathfrak{h}}_{\mathcal{Fib}}^-)$ , the algebra of symmetric polynomials in the commuting variables  $\{\beta_i(-m), | i = 1, 2, 0 < m \in \mathbb{Z}\}$  (see [FLM]), which is a representation of the infinite-dimensional Heisenberg algebra with basis  $\{\mathbb{1}, \beta_i(m) \mid i = 1, 2, m \in \mathbb{Z}\}$  and relations

$$[\beta_i(m), \beta_j(n)] = m(\beta_i, \beta_j)\delta_{m,-n}\mathbb{1} \quad \text{and} \quad [\mathbb{1}, \beta_i(m)] = 0.$$

We refer to  $m$  in the operator  $\beta_i(m)$  as the *mode number* of  $\beta_i(m)$ . We define

$$V_{\mathcal{Fib}} = S(\hat{\mathfrak{h}}_{\mathcal{Fib}}^-) \otimes \mathbb{C}[Q_{\mathcal{Fib}}],$$

where  $\mathbb{C}[Q_{\mathcal{Fib}}]$  is the group algebra of  $Q_{\mathcal{Fib}}$  with basis  $\{e^\beta \mid \beta \in Q_{\mathcal{Fib}}\}$  and multiplication given by  $e^\beta e^\gamma = e^{\beta+\gamma}$ . Note that  $\beta_i(0)$  is central since  $[\beta_i(0), \beta_j(n)] = 0(\beta_i, \beta_j)\delta_{0,-n}\mathbb{1} = 0$ , and there is a diagonal action of  $\beta_i(0)$  on  $\mathbb{C}[Q_{\mathcal{Fib}}]$  by

$$\beta_i(0)e^\lambda = (\beta_i, \lambda)e^\lambda. \tag{6.1}$$

For *homogeneous* vectors in  $V_{\mathcal{Fib}}$ , that is, vectors of the form  $\beta_{i_1}(-m_1)\cdots\beta_{i_r}(-m_r)\mathbb{1} \otimes e^\beta$  where  $0 \leq m_i \in \mathbb{Z}$  and  $i_j \in \{1, 2\}$ ,  $1 \leq j \leq r$ , define the  $\mathbb{Z}$ -valued weight function given by

$$wt\left(\beta_{i_1}(-m_1)\cdots\beta_{i_r}(-m_r)\mathbb{1} \otimes e^\beta\right) = \sum_{j=1}^r m_j + \frac{(\beta, \beta)}{2}. \tag{6.2}$$

As a vector space,  $V_{\mathcal{Fib}}$  is graded by weight:

$$V_{\mathcal{Fib}} = \bigoplus_{m \in \mathbb{Z}} V_{\mathcal{Fib}, m},$$

where  $V_{\mathcal{F}ib,m} = \text{Span}(\{\mathbf{v} \in V_{\mathcal{F}ib} \mid wt(\mathbf{v}) = m\})$ . There is a compatible grading by  $Q_{\mathcal{F}ib}$ ,

$$V_{\mathcal{F}ib} = \bigoplus_{\beta \in Q_{\mathcal{F}ib}} V_{\mathcal{F}ib}^{\beta},$$

where  $V_{\mathcal{F}ib}^{\beta} = S(\hat{\mathfrak{h}}_{\mathcal{F}ib}^{-}) \otimes e^{\beta}$ , that is,  $V_{\mathcal{F}ib,m}^{\beta} = V_{\mathcal{F}ib}^{\beta} \cap V_{\mathcal{F}ib,m}$ . Since the lattice  $Q_{\mathcal{F}ib}$  is indefinite, both  $V_{\mathcal{F}ib}^{\beta}$  and  $V_{\mathcal{F}ib,m}$  are infinite dimensional. However, note that

$$\dim(V_{\mathcal{F}ib,m}^{\beta}) = p\left(m - \frac{(\beta, \beta)}{2}\right) < \infty,$$

where  $p(n)$  is the partition function ([B]).

Later we will give the definition of vertex operators  $Y(\cdot, z) : V_{\mathcal{F}ib} \rightarrow \text{End}(V_{\mathcal{F}ib})[[z, z^{-1}]]$  whose components are given by the notations

$$Y(\mathbf{v}, z) = \sum_{m \in \mathbb{Z}} \mathbf{v}_m z^{-m-1} = \sum_{n \in \mathbb{Z}} Y_n(\mathbf{v}) z^{-n-wt(\mathbf{v})},$$

so

$$\mathbf{v}_{n+wt(\mathbf{v})-1} = Y_n(\mathbf{v}) \in \text{End}(V_{\mathcal{F}ib}).$$

The operator  $\mathbf{v}_m$  is computed by

$$\mathbf{v}_m = \oint Y(\mathbf{v}, z) z^m dz = \text{Res}_{z=0}(Y(\mathbf{v}, z) z^m), \quad (6.3)$$

which is the coefficient of the  $z^{-1}$  term, so

$$Y_n(\mathbf{v}) = \text{Res}_{z=0}(Y(\mathbf{v}, z) z^{n+wt(\mathbf{v})-1}).$$

We define the *vertex operator*  $Y(\mathbf{v}, z)$  for the following choices of  $\mathbf{v} \in V_{\mathcal{F}ib}$ :

For  $\mathbf{v} = \mathbb{1} \otimes e^0$  (called the *vacuum vector*),

$$Y(\mathbb{1} \otimes e^0, z) = I_{V_{\mathcal{F}ib}}.$$

We also have

$$Y(\beta(-1)\mathbb{1} \otimes e^0, z) = \beta(z) = \sum_{n \in \mathbb{Z}} \beta(n) z^{-n-1},$$

and for general  $m \geq 0$ ,

$$Y(\beta(-m-1)\mathbb{1} \otimes e^0, z) = \beta^{(m)}(z) = \frac{1}{m!} \left(\frac{d}{dz}\right)^m \sum_{n \in \mathbb{Z}} \beta(n) z^{-n-1} = \frac{1}{m!} \left(\frac{d}{dz}\right)^m \beta(z).$$

$$Y(\mathbb{1} \otimes e^{\beta}, z) = \exp\left(\sum_{k>0} \frac{\beta(-k)}{k} z^k\right) \exp\left(\sum_{k>0} \frac{\beta(k)}{-k} z^{-k}\right) e^{\beta} z^{\beta(0)} \varepsilon_{\beta},$$

where  $z^{\beta(0)}$  and  $\varepsilon_{\beta}$  act on  $\mathbf{v} = w \otimes e^{\lambda} \in V_{\mathcal{F}ib}^{\lambda}$  by

$$z^{\beta(0)}(w \otimes e^{\lambda}) = z^{(\beta, \lambda)}(w \otimes e^{\lambda}),$$

$$\varepsilon_{\beta}(w \otimes e^{\lambda}) = \varepsilon(\beta, \lambda)(w \otimes e^{\lambda}),$$

and  $\varepsilon(\beta, \lambda) = \pm 1$  is a 2-cocycle (which may be chosen bilinear) subject to the conditions

$$\frac{\varepsilon(\beta, \lambda)}{\varepsilon(\lambda, \beta)} = (-1)^{(\beta, \lambda)}$$

for any  $\beta, \lambda \in Q_{\mathcal{F}ib}$ . We fix such a bilinear 2-cocycle determined by the matrix of values on the simple roots,

$$\left( \varepsilon(\beta_i, \beta_j) \right) = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \quad (6.4)$$

for  $i, j = 1, 2$ . It is clear that for all  $n \in \mathbb{Z}$ ,  $\varepsilon(n\beta, \gamma) = \varepsilon(\beta, n\gamma) = \varepsilon(\beta, \gamma)^n$ .

We sometimes write  $e^\beta = \mathbb{1} \otimes e^\beta$ .

For  $m_{i_1}, \dots, m_{i_r} \geq 0$ , and  $i_1, \dots, i_r = 1, 2$ ,

$$Y\left(\beta_{i_1}(-m_{i_1}-1) \cdots \beta_{i_r}(-m_{i_r}-1) \mathbb{1} \otimes e^\beta, z\right) = : \beta_{i_1}^{(m_1)}(z) \cdots \beta_{i_r}^{(m_r)}(z) Y(\mathbb{1} \otimes e^\beta, z) :,$$

where  $:\cdots:$  is the bosonic normal ordering that places Heisenberg operators with positive mode numbers to the right and negative mode numbers to the left.

It can be shown that  $(V_{\mathcal{F}ib}, Y(\cdot, z), \omega = \omega_{\mathcal{F}ib}, \mathbb{1} \otimes e^0)$  is a vertex algebra, where

$$\omega = \frac{1}{2} \sum_{i=1}^2 \beta_i(-1) \lambda_i(-1) \mathbb{1} \otimes e^0$$

(so  $wt(\omega) = 2$ ), and  $\{\lambda_1, \lambda_2\}$  are the fundamental weights of  $\mathcal{F}ib$  (see Chapter 2). We call  $\omega$  the *conformal vector*, and

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = \sum_{n \in \mathbb{Z}} Y_n(\omega) z^{-n-2} = \sum_{m \in \mathbb{Z}} \omega_m z^{-m-1},$$

where the operators  $L_n$  represent the Virasoro algebra with central charge  $c_{vir} = \text{rank}(Q_{\mathcal{F}ib}) = 2$  and Lie brackets given by

$$[L_m, L_n] = (m-n)L_{m+n} + c_{vir} \delta_{m,-n} (m^3 - m)/12 \quad \text{and} \quad [c_{vir}, L_n] = 0.$$

It immediately follows that  $L_n = \omega_{n+1}$ . The Virasoro operators act only on the Fock space component of each vector. The operator  $L_0$  acts diagonalizably on  $V_{\mathcal{F}ib}$ , such that for  $\mathbf{v} \in V_{\mathcal{F}ib, m}$ ,  $L_0 \cdot \mathbf{v} = m\mathbf{v} = wt(\mathbf{v})\mathbf{v}$ . We also have that  $L_{-1} = \omega_0$  acts as a derivation,

$$[L_{-1}, Y(\mathbf{u}, z)] = Y(L_{-1}\mathbf{u}, z) = \frac{d}{dz} Y(\mathbf{u}, z). \quad (6.5)$$

which will prove useful in Section 6.4.

**Lemma 6.1** ([FLM]). *Let  $V$  be a vertex algebra constructed from root lattice  $Q$  with Virasoro operators  $L_i$  and vacuum vector  $\mathbb{1}$ . Then for  $\alpha \in Q$ ,*

$$(i) \quad [L_m, \alpha(-n)] = n\alpha(m-n) \text{ for } m, n \in \mathbb{Z},$$



(ii)  $L_m \mathbb{1} = 0$  for  $0 \leq m \in \mathbb{Z}$ .

If  $\mathbf{u}, \mathbf{v} \in V$  then the weight of vector  $\mathbf{u}_n(\mathbf{v})$  is given by the formula

$$wt(\mathbf{u}_n(\mathbf{v})) = wt(\mathbf{u}) + wt(\mathbf{v}) - n - 1,$$

which shows that if  $\mathbf{u} \in V_1$  and  $\mathbf{v} \in V_m$ ,  $wt(\mathbf{u}_0(\mathbf{v})) = 1 + m - 0 - 1 = m$ , so  $\mathbf{u}_0(V_{\mathcal{F}ib,m}) \subset V_{\mathcal{F}ib,m}$ .

## 6.2 The vertex algebra $V_{\mathcal{F}}$

We now apply Borcherd's method to the root lattice  $Q_{\mathcal{F}} = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3$ . Similar to the previous construction we have a Fock space  $S(\hat{\mathfrak{h}}_{\mathcal{F}}^-)$ , the algebra of symmetric polynomials in the commuting variables  $\{\alpha_i(m) \mid 0 < m \in \mathbb{Z}, 1 \leq i \leq 3\}$ , which is a representation of the Heisenberg algebra with basis  $\{\mathbb{1}, \alpha_i(m) \mid m \in \mathbb{Z}, 1 \leq i \leq 3\}$ , and relations

$$[\alpha_i(m), \alpha_j(n)] = m(\alpha_i, \alpha_j)\delta_{m,-n}\mathbb{1}, \quad \text{and} \quad [\mathbb{1}, \alpha_i(m)] = 0.$$

We define

$$V_{\mathcal{F}} = S(\hat{\mathfrak{h}}_{\mathcal{F}}^-) \otimes \mathbb{C}[Q_{\mathcal{F}}].$$

The group algebra  $\mathbb{C}[Q_{\mathcal{F}}]$ ,  $\mathbb{Z}$ -valued weight function  $wt$  on homogeneous vectors in  $V_{\mathcal{F}}$ , gradings of the vector space  $V_{\mathcal{F}}$ , vertex operators  $Y(\cdot, z) : V_{\mathcal{F}} \rightarrow \text{End}(V_{\mathcal{F}})[[z, z^{-1}]]$ , are also defined as before with a new choice of 2-cocycle (see below).

We thus have the vertex algebra  $(V_{\mathcal{F}}, Y(\cdot, z), \omega_{\mathcal{F}}, \mathbb{1} \otimes e^0)$  where

$$\omega_{\mathcal{F}} = \frac{1}{2} \sum_{i=1}^3 \alpha_i(-1) \omega_i(-1) \mathbb{1} \otimes e^0,$$

is the conformal vector for  $V_{\mathcal{F}}$ , and  $\{\omega_1, \omega_2, \omega_3\}$  are the fundamental weights for  $\mathcal{F}$  (see Section 1.3). Since  $Q_{\mathcal{F}ib} \subset Q_{\mathcal{F}}$ , we get  $V_{\mathcal{F}ib} \subset V_{\mathcal{F}}$ , where compatibility of the vertex algebra structures requires that the choice of 2-cocycle for  $\mathcal{F}ib$  be the restriction to  $Q_{\mathcal{F}ib}$  of the 2-cocycle of  $\mathcal{F}$ . We fix such a bilinear 2-cocycle, determined by the matrix of values on the simple roots,

$$\left( \varepsilon(\alpha_i, \alpha_j) \right) = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}. \quad (6.6)$$

Note that the 2-cocycles for  $\mathcal{F}ib$  and  $\mathcal{F}$  can be distinguished by their domains, so we use the same notation for both. Using the notation

$$\varepsilon_{ij} = \varepsilon(\alpha_i, \alpha_j),$$

we have

$$\varepsilon(\beta_1, \beta_1) = \varepsilon(2\alpha_1 + \alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 + \alpha_3) = \varepsilon_{11}^4 \varepsilon_{12}^2 \varepsilon_{13}^2 \varepsilon_{21}^2 \varepsilon_{22} \varepsilon_{23} \varepsilon_{31}^2 \varepsilon_{32} \varepsilon_{33} = -1,$$

$$\varepsilon(\beta_1, \beta_2) = \varepsilon(2\alpha_1 + \alpha_2 + \alpha_3, \alpha_2) = \varepsilon_{12}^2 \varepsilon_{22} \varepsilon_{32} = 1,$$

$$\varepsilon(\beta_2, \beta_1) = \varepsilon(\alpha_2, 2\alpha_1 + \alpha_2 + \alpha_3) = \varepsilon_{21}^2 \varepsilon_{22} \varepsilon_{23} = -1,$$

$$\varepsilon(\beta_2, \beta_2) = \varepsilon(\alpha_2, \alpha_2) = \varepsilon_{22} = -1,$$

which shows that the 2-cocycles for  $V_{\mathcal{F}}$  and  $V_{\mathcal{F}ib}$  are compatible.

### 6.3 Representation of $\mathcal{Fib}$ acting on $V_{\mathcal{F}}$

In [FLM], it was shown that if  $V_L$  is a vertex operator algebra constructed from a positive-definite even lattice  $L$ , then the subspace of  $End(V_L)$  spanned by the operators  $\{\mathbf{u}_m \mid \mathbf{u} \in V_L, m \in \mathbb{Z}\}$  is a Lie subalgebra. In [B], Borchers showed that this result extends to indefinite lattices, and in both cases, one has the commutator identity

$$[\mathbf{u}_m, \mathbf{v}_n] = \sum_{i \in \mathbb{N}} \binom{m}{i} (\mathbf{u}_i(\mathbf{v}))_{m+n-i}.$$

In special cases we have

$$[\mathbf{u}_0, \mathbf{v}_n] = (\mathbf{u}_0(\mathbf{v}))_n \quad \text{and} \quad [\mathbf{u}_0, \mathbf{v}_0] = (\mathbf{u}_0(\mathbf{v}))_0,$$

so the space  $End(V_L)_0$  of operators spanned by  $\{\mathbf{u}_0 \mid \mathbf{u} \in V_L\}$  is a Lie subalgebra of  $End(V_L)$ , and the space  $End(V_L)_n$  spanned by  $\{\mathbf{v}_n \mid \mathbf{v} \in V_L\}$  is a module for  $End(V_L)_0$ . Furthermore, the span of  $\{\mathbf{u}_0 \mid \mathbf{u} \in V_L, wt(\mathbf{u}) = 1\}$  is a Lie subalgebra of  $End(V_L)_0$  since  $wt(\mathbf{u}_0(\mathbf{v})) = wt(\mathbf{u}) + wt(\mathbf{v}) - 0 - 1 = 1$ . We focus on the following special weight-1 vectors.

**Definition 6.2.** Let  $\mathcal{L}$  be a KM algebra,  $V_{\mathcal{L}}$  the vertex algebra constructed from the root lattice of  $\mathcal{L}$ , with Virasoro operators  $L_n^{\mathcal{L}}, n \in \mathbb{Z}$ . For  $i \in \mathbb{Z}$ , the **physical  $i$ -space** of  $V_{\mathcal{L}}$  is

$$P_i^{\mathcal{L}} = \{\mathbf{v} \in V_{\mathcal{L}} \mid L_0^{\mathcal{L}} \mathbf{v} = i\mathbf{v}, L_m^{\mathcal{L}} \mathbf{v} = 0 \text{ if } m > 0\}.$$

From now on the reader may assume that  $P_n$  means  $P_n^{\mathcal{F}}$  and  $L_n$  means  $L_n^{\mathcal{F}}$ . If a vector  $\mathbf{v} \in P_1$ , then  $L_0 \mathbf{v} = wt(\mathbf{v})\mathbf{v} = \mathbf{v}$ , so  $P_1 \subset V_{\mathcal{F},1}$ . For  $wt(\mathbf{v})$  arbitrary and any  $n \in \mathbb{Z}$  we have

$$wt(L_n(\mathbf{v})) = wt(\omega_{n+1}(\mathbf{v})) = wt(\omega) + wt(\mathbf{v}) - (n+1) - 1 = wt(\mathbf{v}) - n,$$

and in particular, if  $wt(\mathbf{v}) = 0$ , then  $wt(L_{-1}(\mathbf{v})) = 1$ , so  $L_{-1}(P_0) \subset P_1$ .

By a result of Borchers, the quotient space  $\overline{P}_1 = P_1 / L_{-1}(P_0)$  is a Lie algebra, where the Lie algebra bracket is given by

$$[\overline{\mathbf{u}}, \overline{\mathbf{v}}] = \overline{\mathbf{u}_0(\mathbf{v})} \quad \text{for} \quad \overline{\mathbf{u}}, \overline{\mathbf{v}} \in \overline{P}_1 \tag{6.7}$$

([B] §5). This formula, which defines the adjoint representation of  $\overline{P}_1$ , also defines the action of  $\overline{P}_1$  on itself as a  $\overline{P}_1$ -module. In general,  $V_{\mathcal{F}}$  is also a  $\overline{P}_1$ -module with the action  $\overline{\mathbf{u}} \cdot \mathbf{v} = \overline{\mathbf{u}_0(\mathbf{v})}$  for  $\mathbf{v} \in V_{\mathcal{F}}$ . Note that if  $\mathbf{u} \in L_{-1}(P_0)$ , then  $\overline{\mathbf{u}}_0 = \overline{\mathbf{0}}_0 = 0$ .

Borcherds gives us the Lie algebra representation  $\pi_{\mathcal{F}} : \mathcal{F} \rightarrow \overline{P}_1$  defined by

$$\pi_{\mathcal{F}}(e_i) = \mathbb{1} \otimes e^{\alpha_i}, \quad \pi_{\mathcal{F}}(f_i) = -\mathbb{1} \otimes e^{-\alpha_i}, \quad \text{and} \quad \pi_{\mathcal{F}}(h_i) = \alpha_i(-1) \mathbb{1} \otimes e^0$$

for  $i = 1, 2, 3$  (for convenience we sometimes suppress the bars for vectors in  $\overline{P}_1$ ). In Appendix B.2, we verify that these elements indeed satisfy the Serre relations of  $\mathcal{F}$ . He also gives us  $\pi_{\mathcal{F}ib} : \mathcal{F}ib \rightarrow \overline{P}_1^{\mathcal{F}ib} = P_1^{\mathcal{F}ib} / L_{-1}^{\mathcal{F}ib}(P_0^{\mathcal{F}ib})$  given by

$$\pi_{\mathcal{F}ib}(E_i) = \mathbb{1} \otimes e^{\beta_i}, \quad \pi_{\mathcal{F}ib}(F_i) = -\mathbb{1} \otimes e^{-\beta_i}, \quad \text{and} \quad \pi_{\mathcal{F}ib}(H_i) = \beta_i(-1) \mathbb{1} \otimes e^0,$$

for  $i = 1, 2$  (we have again suppressed the bar notation, this time for vectors in  $\overline{P}_1^{\mathcal{F}ib}$ ).

We also prove in Appendix B.3 that restricting  $\pi_{\mathcal{F}}$  to  $\mathcal{F}ib$  gives a representation  $\pi_{\mathcal{F}}|_{\mathcal{F}ib}$  of  $\mathcal{F}ib$  acting on  $V_{\mathcal{F}}$  that is compatible with  $\mathcal{F}ib \subset \mathcal{F}$ , i.e.,  $\pi_{\mathcal{F}}|_{\mathcal{F}ib} = \pi_{\mathcal{F}ib}$ . This is done by showing for  $i = 1, 2$ ,

$$\pi_{\mathcal{F}}(E_i) = \mathbb{1} \otimes e^{\beta_i}, \quad \pi_{\mathcal{F}}(F_i) = -\mathbb{1} \otimes e^{-\beta_i}, \quad \text{and} \quad \pi_{\mathcal{F}}(H_i) = \beta_i(-1) \mathbb{1} \otimes e^0,$$

where  $E_1 = \frac{1}{2}e_{1123}$ ,  $E_2 = e_2$ ,  $F_1 = -\frac{1}{2}f_{1123}$ ,  $F_2 = f_2$ ,  $H_1 = 2h_1 + h_2 + h_3$ , and  $H_2 = h_2$ .

The action of  $\pi_{\mathcal{F}}(\mathcal{F})$  on  $\overline{P}_1$  is defined by the adjoint representation of  $\overline{P}_1$  and the bracket formula (6.7), and is compatible with the action of  $\mathcal{F}$  on itself, since for all  $x, y \in \mathcal{F}$ ,

$$\pi_{\mathcal{F}}(x) \cdot \pi_{\mathcal{F}}(y) = [\pi_{\mathcal{F}}(x), \pi_{\mathcal{F}}(y)] = \pi_{\mathcal{F}}([x, y]) = \pi_{\mathcal{F}}(x \cdot y),$$

where the action on the left-hand side is on  $\overline{P}_1$  and the right-hand side is on  $\mathcal{F}$ .

The restricted representation  $\pi_{\mathcal{F}ib}$  makes  $\overline{P}_1$  also a  $\mathcal{F}ib$ -module, where the action of  $X \in \mathcal{F}ib$  on  $\overline{v} \in \overline{P}_1$  is given by

$$\pi_{\mathcal{F}ib}(X) \cdot \overline{v} = (\pi_{\mathcal{F}ib}(X))_0(\overline{v}). \quad (6.8)$$

## 6.4 Finding lowest-weight vectors for $\mathcal{F}ib$ in $\overline{P}_1^{-\rho}$

We look for lowest-weight vectors for  $\mathcal{F}ib$  in  $\overline{P}_1$ , starting with the weight space  $\overline{P}_1^{-\rho}$  where  $-\rho = -\rho_{\mathcal{F}ib} = \beta_1 + \beta_2$ . In Theorem 4.5 we found that the space of lowest-weight vectors  $Low_{\mathcal{F}ib(0)}(-\rho)$  has dimension 1, and has basis  $(v_{-\rho} = -3e_{12123} + 2E_{21})$ . Then

$$\overline{x} = \pi_{\mathcal{F}}(v_{-\rho}) = \pi_{\mathcal{F}}(-3e_{12123} + 2E_{21}) \in \overline{P}_1^{-\rho}$$

is also a lowest-weight vector for  $\mathcal{F}ib$ , since for  $i = 1, 2$ ,

$$\pi_{\mathcal{F}ib}(F_i) \cdot \overline{x} = [\pi_{\mathcal{F}ib}(F_i), \overline{x}] = [\pi_{\mathcal{F}}(F_i), \pi_{\mathcal{F}}(v_{-\rho})] = \pi_{\mathcal{F}}([F_i, v_{-\rho}]) = \overline{0}.$$

Therefore  $\overline{0} \neq \overline{x} \in Low_{\overline{P}_1}(-\rho)$ , however, we will now show that  $\dim Low_{\overline{P}_1}(-\rho) > 1$ .

**Theorem 6.3.** *A basis for  $\bar{P}_1^{-\rho}$  is  $B_{\bar{P}_1}^{-\rho} = (\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_2, \bar{\mathbf{p}}_3)$ , where*

$$\begin{aligned}\mathbf{p}_1 &= \left( \alpha_2(-2) + \alpha_2(-1)^2 \right) \mathbb{1} \otimes e^{-\rho}, \\ \mathbf{p}_2 &= \left( -\alpha_1(-1)^2 + \alpha_3(-1)^2 \right) \mathbb{1} \otimes e^{-\rho}, \text{ and} \\ \mathbf{p}_3 &= \alpha_1(-1)\alpha_3(-1) \mathbb{1} \otimes e^{-\rho}.\end{aligned}$$

*Proof.* All vectors in  $\bar{P}_1^{-\rho}$  are weight 1 and have a degree-2 Fock space polynomial, since  $\|-\rho\|^2 = -2$  (cf. equation (6.2)). We have the ordered basis for the subspace of degree-2 polynomials in  $S(\hat{\mathfrak{h}}_{\mathcal{F}}^-)$ ,

$$\begin{aligned}S = (u_i \mid 1 \leq i \leq 9) &= \left( \alpha_1(-2), \alpha_2(-2), \alpha_3(-2), \alpha_1(-1)^2, \alpha_2(-1)^2, \alpha_3(-1)^2, \right. \\ &\quad \left. \alpha_1(-1)\alpha_2(-1), \alpha_2(-1)\alpha_3(-1), \alpha_1(-1)\alpha_3(-1) \right),\end{aligned}$$

which gives us the basis for the 9-dimensional space  $V_{\mathcal{F},1}^{-\rho}$ ,

$$B_{\mathcal{F},1}^{-\rho} = \left( u_i \mathbb{1} \otimes e^{-\rho} \mid 1 \leq i \leq 9 \right).$$

Let

$$\mathbf{v} = \left( \sum_{i=1}^9 c_i u_i \right) \mathbb{1} \otimes e^{-\rho}, \text{ where } c_i \in \mathbb{C}$$

be a general vector in  $V_{\mathcal{F},1}^{-\rho}$ . Then  $\mathbf{v} \in P_1^{-\rho}$  if and only if  $L_m(\mathbf{v}) = \mathbf{0}$  for  $m = 1, 2$ . Note that  $L_0(\mathbf{v}) = \mathbf{v}$  is immediately true since  $wt(\mathbf{v}) = 1$ . We therefore find  $P_1^{-\rho} = \text{Ker}(L_1) \cap \text{Ker}(L_2)$ . Using Proposition 6.1, we have for  $i = 1, 2$  and  $j = 1, 2, 3$ ,

$$\begin{aligned}L_i(\alpha_j(-2) \mathbb{1} \otimes e^{-\rho}) &= \left( [L_i, \alpha_j(-2)] + \alpha_j(-2)L_i \right) \mathbb{1} \otimes e^{-\rho} \\ &= 2\alpha_j(i-2) \mathbb{1} \otimes e^{-\rho},\end{aligned}$$

so

$$L_1(\alpha_j(-2) \mathbb{1} \otimes e^{-\rho}) = 2\alpha_j(-1) \mathbb{1} \otimes e^{-\rho} \quad \text{and} \quad L_2(\alpha_j(-2) \mathbb{1} \otimes e^{-\rho}) = -2\delta_{j2} \mathbb{1} \otimes e^{-\rho}.$$

For  $(j, k) = (1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)$ , we have

$$\begin{aligned}L_i(\alpha_j(-1)\alpha_k(-1) \mathbb{1} \otimes e^{-\rho}) &= \left( [L_i, \alpha_j(-1)] + \alpha_j(-1)L_i \right) \alpha_k(-1) \mathbb{1} \otimes e^{-\rho} \\ &= \left( \alpha_j(i-1)\alpha_k(-1) + \alpha_j(-1)\alpha_k(i-1) \right) \mathbb{1} \otimes e^{-\rho}.\end{aligned}$$

We now have formulas for computing  $L_i(u_t \mathbb{1} \otimes e^{-\rho})$  for  $u_t \mathbb{1} \otimes e^{-\rho} \in \mathcal{B}_{\mathcal{F},1}^{-\rho}$ ,  $1 \leq t \leq 9$ , and  $i = 1, 2$ :

$$\begin{aligned}L_1(\alpha_j(-1)\alpha_k(-1) \mathbb{1} \otimes e^{-\rho}) &= -\delta_{j2}\alpha_k(-1) - \delta_{k2}\alpha_j(-1) \mathbb{1} \otimes e^{-\rho}, \\ L_1(\alpha_j(-1)^2 \mathbb{1} \otimes e^{-\rho}) &= -2\delta_{j2}\alpha_j(-1) \mathbb{1} \otimes e^{-\rho},\end{aligned}$$

$$L_2(\alpha_j(-1)\alpha_k(-1)\mathbb{1} \otimes e^{-\rho}) = (\alpha_j, \alpha_k)\mathbb{1} \otimes e^{-\rho}, \quad \text{and}$$

$$L_2(\alpha_j(-1)^2\mathbb{1} \otimes e^{-\rho}) = (\alpha_j, \alpha_j)\mathbb{1} \otimes e^{-\rho}.$$

Applying these formulas we have

$$L_1(\mathbf{v}) = (\alpha_1(-1)(2c_1 - c_7) + \alpha_2(-1)(2c_2 - 2c_5) + \alpha_3(-1)(2c_3 - c_8))\mathbb{1} \otimes e^{-\rho} = \mathbf{0}$$

and

$$L_2(\mathbf{v}) = (-2c_2 + 2c_4 + 2c_5 + 2c_6 - 2c_7 - c_8)\mathbb{1} \otimes e^{-\rho} = \mathbf{0}.$$

The resulting system of linear equations

$$\begin{cases} 2c_1 & & & -c_7 & = 0 \\ & 2c_2 & & -2c_5 & = 0 \\ & & 2c_3 & & -c_8 = 0 \\ & -2c_2 & +2c_4 + 2c_5 + 2c_6 - 2c_7 - c_8 & = 0 \end{cases}$$

has a 5-dimensional solution space  $P_1^{-\rho} = \text{Ker}(L_1) \cap \text{Ker}(L_2) = \text{Span}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5)$ , where

$$\mathbf{p}_1 = (\alpha_2(-2) + \alpha_2(-1)^2)\mathbb{1} \otimes e^{-\rho} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\mathbf{p}_2 = (-\alpha_1(-1)^2 + \alpha_3(-1)^2)\mathbb{1} \otimes e^{-\rho} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\mathbf{p}_3 = \alpha_1(-1)\alpha_3(-1)\mathbb{1} \otimes e^{-\rho} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T,$$

$$\mathbf{p}_4 = \left(\frac{1}{2}\alpha_1(-2) + \alpha_1(-1)^2\right)\mathbb{1} \otimes e^{-\rho} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T,$$

$$\mathbf{p}_5 = \left(\frac{1}{2}\alpha_3(-2) + \frac{1}{2}\alpha_1(-1)^2 + \alpha_2(-1)\alpha_3(-1)\right)\mathbb{1} \otimes e^{-\rho} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T,$$

where vectors are written with respect to the basis  $B_{\mathcal{F},1}^{-\rho}$ .

Let  $Q : P_1^{-\rho} \rightarrow \overline{P}_1^{-\rho}$  be the quotient map given by  $Q(\mathbf{v}) = \overline{\mathbf{v}}$  where  $\overline{\mathbf{v}}$  is the equivalence class of  $\mathbf{v}$  in the quotient space  $\overline{P}_1^{-\rho} = P_1^{-\rho} / L_{-1}(P_0^{-\rho})$ , so if  $\mathbf{v} \in L_{-1}(P_0^{-\rho})$  then  $Q(\mathbf{v}) = \overline{\mathbf{0}}$ . We find a basis for  $\text{Ker}(Q) \subset P_1^{-\rho}$ . Let  $\mathbf{v} = L_{-1}(\mathbf{u})$  where  $\mathbf{u} \in P_0^{-\rho}$ . Then  $\mathbf{v}_0 = 0$ , so by equations (6.3) and (6.5),

$$\mathbf{v}_0 = (L_{-1}(\mathbf{u}))_0 = \oint Y(L_{-1}(\mathbf{u}), z) dz = \oint \frac{d}{dz} Y(\mathbf{u}, z) dz = 0.$$

Note that since  $wt(\mathbf{u}) = 0$  and  $(-\rho, -\rho) = -2$ , the Fock space component of  $\mathbf{u}$  has degree 1. Writing  $\mathbf{u} = \alpha(-1) \otimes e^{-\rho}$  we have

$$\begin{aligned} 0 &= \oint \frac{d}{dz} Y(\alpha(-1) \otimes e^{-\rho}, z) dz \\ &= \oint \frac{d}{dz} : \alpha^{(0)}(z) Y(e^{-\rho}, z) : dz \end{aligned}$$

$$\begin{aligned}
 &= \oint : \alpha^{(1)}(z)Y(e^{-\rho}, z) : - : \alpha^{(0)}(z)\rho^{(0)}(z)Y(e^{-\rho}, z) : dz \\
 &= \oint Y(\alpha(-2)\mathbb{1} \otimes e^{-\rho}, z) - Y(\alpha(-1)\rho(-1)\mathbb{1} \otimes e^{-\rho}, z) dz \\
 &= \oint Y((\alpha(-2) - \alpha(-1)\rho(-1))\mathbb{1} \otimes e^{-\rho}, z) dz,
 \end{aligned}$$

which holds if

$$\left( (\alpha(-2) - \alpha(-1)\rho(-1))\mathbb{1} \otimes e^{-\rho} \right)_0 = 0, \quad (6.9)$$

or equivalently, if  $(\alpha(-2) - \alpha(-1)\rho(-1))\mathbb{1} \otimes e^{-\rho} \in \text{Ker}(Q)$ . Furthermore, observe that Lemma 6.1 and equation (6.1) give us

$$L_2(\alpha(-1)\mathbb{1} \otimes e^{-\rho}) = ([L_2, \alpha(-1)] + \alpha(-1)L_2)\mathbb{1} \otimes e^{-\rho} = 0$$

and

$$L_1(\alpha(-1)\mathbb{1} \otimes e^{-\rho}) = ([L_1, \alpha(-1)] + \alpha(-1)L_1)\mathbb{1} \otimes e^{-\rho} = \alpha(0)\mathbb{1} \otimes e^{-\rho} = (\alpha, -\rho)\mathbb{1} \otimes e^{-\rho}.$$

Thus if  $u = (a\alpha_1(-1) + b\alpha_2(-1) + c\alpha_3(-1))\mathbb{1} \otimes e^{-\rho} \in P_0^{-\rho}$  where  $a, b, c \in \mathbb{C}$ , we have

$$L_1(\mathbf{u}) = (a(\cancel{\alpha_1, -\rho}) + b(\alpha_2, -\rho) + c(\cancel{\alpha_3, -\rho}))\mathbb{1} \otimes e^{-\rho} = -b\mathbb{1} \otimes e^{-\rho} = \mathbf{0},$$

hence  $b = 0$  and  $\mathbf{u} \in \text{Span}(\alpha_1(-1)\mathbb{1} \otimes e^{-\rho}, \alpha_3(-1)\mathbb{1} \otimes e^{-\rho})$ . Substituting  $\alpha = c_1\alpha_1 + c_2\alpha_3$  for some  $c_1, c_2 \in \mathbb{C}$  and  $-\rho = 2\alpha_1 + 2\alpha_2 + \alpha_3$  into equation (6.9) gives us

$$\begin{aligned}
 0 &= \left( (c_1\alpha_1(-2) + c_2\alpha_3(-1) + (c_1\alpha_1(-1) + c_2\alpha_3(-1)) \right. \\
 &\quad \left. \cdot (2\alpha_1 + 2\alpha_2 + \alpha_3)(-1))\mathbb{1} \otimes e^{-\rho} \right)_0 \\
 &= \left( (c_1(\alpha_1(-2) + 2\alpha_1(-1)^2 + 2\alpha_1(-1)\alpha_2(-1) + \alpha_1(-1)\alpha_3(-1)) \right. \\
 &\quad \left. + c_2(\alpha_3(-2) + 2\alpha_2(-1)\alpha_3(-1) + \alpha_3(-1)^2 + 2\alpha_1(-1)\alpha_3(-1)))\mathbb{1} \otimes e^{-\rho} \right)_0.
 \end{aligned}$$

Thus we have  $L_{-1}(P_0)^{-\rho} = \text{Ker}(Q) = \text{Span}(\mathbf{q}_1, \mathbf{q}_2)$ , where

$$\begin{aligned}
 \mathbf{q}_1 &= (\alpha_1(-2) + 2\alpha_1(-1)^2 + 2\alpha_1(-1)\alpha_2(-1) + \alpha_1(-1)\alpha_3(-1))\mathbb{1} \otimes e^{-\rho} \\
 &= \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 1 \end{bmatrix}^T, \\
 \mathbf{q}_2 &= (\alpha_3(-2) + 2\alpha_2(-1)\alpha_3(-1) + \alpha_3(-1)^2 + 2\alpha_1(-1)\alpha_3(-1))\mathbb{1} \otimes e^{-\rho} \\
 &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 2 \end{bmatrix}^T.
 \end{aligned}$$

where vectors are written with respect to the basis  $B_{\mathcal{F},1}^{-\rho}$ . Observe that for each  $i = 1, 2$ ,  $\bar{\mathbf{q}}_i = \bar{\mathbf{0}}$  gives a congruence relation on the vectors in  $P_1^{-\rho}$ .

We now compute  $\overline{P}_1^{-\rho} = Ker(Q) \cap Ker(L_1) \cap Ker(L_2)$  by finding the null space of

$$\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \mathbf{p}_4 & \mathbf{p}_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 2 & 0 & 0 & -1 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The solution gives the dependence relations:

$$\mathbf{p}_4 = \frac{1}{2}(\mathbf{q}_1 - \mathbf{p}_3) \quad \text{and} \quad \mathbf{p}_5 = \frac{1}{2}\mathbf{q}_2 - \frac{1}{2}\mathbf{p}_2 - \mathbf{p}_3,$$

so  $2\overline{\mathbf{p}}_4 = \overline{\mathbf{p}}_3$  and  $2\overline{\mathbf{p}}_5 = -\overline{\mathbf{p}}_2 - 2\overline{\mathbf{p}}_3$ . Thus  $\overline{P}_1^{-\rho}$  is 3-dimensional, with basis

$$B_{\overline{P}_1}^{-\rho} = (\overline{\mathbf{p}}_1, \overline{\mathbf{p}}_2, \overline{\mathbf{p}}_3).$$

□

The theorem demonstrates that  $\pi_{\mathcal{F}}$  is not surjective, since

$$Mult_{\mathcal{F}}(-\rho) = 2 < 3 = Mult_{\overline{P}_1}(-\rho).$$

**Theorem 6.4.** *A basis for  $Low_{\overline{P}_1}(-\rho)$  is  $B_{Low}^{-\rho} = (\overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2)$ , where*

$$\overline{\mathbf{x}}_1 = \overline{\mathbf{p}}_1 + 2\overline{\mathbf{p}}_2,$$

$$\overline{\mathbf{x}}_2 = -\overline{\mathbf{p}}_1 + 3\overline{\mathbf{p}}_3.$$

*Proof.* If  $\overline{\mathbf{x}} \in Low_{\overline{P}_1}(-\rho)$  then  $F_i \cdot \overline{\mathbf{x}} = \overline{\mathbf{0}}$  for  $i = 1, 2$ , so we compute  $Ker(F_1) \cap Ker(F_2) \subset \overline{P}_1^{-\rho}$ . As before, we find general formulas for  $F_i \cdot u_t \mathbb{1} \otimes e^{-\rho}$  where  $u_t \mathbb{1} \otimes e^{-\rho} \in B_{\mathcal{F},1}^{-\rho}$ ,  $1 \leq t \leq 9$ , to help compute the action of  $F_i$  on the basis elements  $\overline{\mathbf{p}}_j \in B_{\overline{P}_1}^{-\rho}$ . We also make use of Lemma B.3 in Appendix B.

For  $i = 1, 2$  and  $j = 1, 2, 3$ , we have

$$\begin{aligned} F_i \cdot \alpha_j(-2) \mathbb{1} \otimes e^{\beta_1 + \beta_2} &= e_0^{-\beta_i}(\alpha_j(-2) \mathbb{1} \otimes e^{\beta_1 + \beta_2}) \\ &= Res_{z=0}(Y(e^{-\beta_i}, z))\alpha_j(-2) \mathbb{1} \otimes e^{\beta_1 + \beta_2} \\ &= Res_{z=0}\left(\exp\left(\sum_{k>0} \frac{-\beta_i(-k)}{k} z^k\right) \exp\left(\sum_{k>0} \frac{-\beta_i(k)}{-k} z^{-k}\right) e^{-\beta_i} z^{-\beta_i(0)} \varepsilon_{-\beta_i}\right) \alpha_j(-2) \mathbb{1} \otimes e^{\beta_1 + \beta_2}. \end{aligned}$$

Note that equation (6.4) gives us

$$\varepsilon(-\beta_i, \beta_1 + \beta_2) = \varepsilon(\beta_i, \beta_1)^{-1} \varepsilon(\beta_i, \beta_2)^{-1} = (-1)^i \quad \text{and} \quad z^{-\beta_i(0)} e^{\beta_1 + \beta_2} = z^{(-\beta_i, \beta_1 + \beta_2)} = z$$

for  $i = 1, 2$ . Also, we observe that any operator in the second formal series above whose mode number or polynomial degree is greater than 2 will annihilate  $\alpha_j(-2)\mathbb{1}$ . The surviving terms of this formal series are shown below in the continuation of the computation:

$$\begin{aligned} F_i \cdot \alpha_j(-2)\mathbb{1} \otimes e^{\beta_1 + \beta_2} &= (-1)^i \text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\beta_i(-k)}{k} z^k \right) \left( I + \frac{1}{1!} \left( \frac{\beta_i(1)}{1} z^{-1} + \frac{\beta_i(2)}{2} z^{-2} \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{2!} \left( \frac{\beta_i(1)}{1} z^{-1} \right)^2 \right) z \right) \alpha_j(-2)\mathbb{1} \otimes e^{\beta_{3-i}} \\ &= (-1)^i \text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\beta_i(-k)}{k} z^k \right) \left( \alpha_j(-2)z + \frac{\beta_i(1)}{2} \alpha_j(-2)z^{-1} + \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \beta_i(1)^2 \alpha_j(-2)z^{-1} \right) \mathbb{1} \otimes e^{\beta_{3-i}} \right) \\ &= (-1)^i \text{Res}_{z=0} \left( \exp(I + h.o.t.) \left( \alpha_j(-2)z + (\beta_i, \alpha_j)z^{-1} \right) \mathbb{1} \otimes e^{\beta_{3-i}} \right) \\ &= (-1)^i (\beta_i, \alpha_j) \mathbb{1} \otimes e^{\beta_{3-i}}. \end{aligned}$$

The remaining basis vectors of  $B_{\mathcal{F},1}^{-\rho}$  are of the form  $\alpha_j(-1)\alpha_k(-1)\mathbb{1} \otimes e^{\beta_1 + \beta_2}$  for  $1 \leq j, k \leq 3$ , so we compute  $F_i \cdot \alpha_j(-1)\alpha_k(-1)\mathbb{1} \otimes e^{\beta_1 + \beta_2}$ :

$$\begin{aligned} F_i \cdot \alpha_j(-1)\alpha_k(-1)\mathbb{1} \otimes e^{\beta_1 + \beta_2} &= e_0^{-\beta_i} \left( \alpha_j(-1)\alpha_k(-1)\mathbb{1} \otimes e^{\beta_1 + \beta_2} \right) \\ &= \text{Res}_{z=0} \left( Y(e^{-\beta_i}, z) \right) \alpha_j(-1)\alpha_k(-1)\mathbb{1} \otimes e^{\beta_1 + \beta_2} \\ &= \text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\beta_i(-k)}{k} z^k \right) \exp \left( \sum_{k>0} \frac{-\beta_i(k)}{-k} z^{-k} \right) e^{-\beta_i} z^{-\beta_i(0)} \varepsilon_{-\beta_i} \right) \\ &\quad \cdot \alpha_j(-1)\alpha_k(-1)\mathbb{1} \otimes e^{\beta_1 + \beta_2}. \end{aligned}$$

Now any operator in the second formal series above whose mode number is greater than 1 or whose polynomial degree is greater than 2 will annihilate  $\alpha_j(-1)\alpha_k(-1)\mathbb{1}$ . Continuing the computation,

$$\begin{aligned} F_i \cdot \alpha_j(-1)\alpha_k(-1)\mathbb{1} \otimes e^{\beta_1 + \beta_2} &= (-1)^i \text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\beta_i(-k)}{k} z^k \right) \right. \\ &\quad \cdot \left( I + \frac{1}{1!} \left( \frac{\beta_i(1)}{1} z^{-1} + \frac{1}{2!} \left( \frac{\beta_i(1)}{1} z^{-1} \right)^2 \right) z \right) \alpha_j(-1)\alpha_k(-1)\mathbb{1} \otimes e^{\beta_{3-i}} \\ &= (-1)^i \text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\beta_i(-k)}{k} z^k \right) \left( \alpha_j(-1)\alpha_k(-1)z + \beta_i(1)\alpha_j(-1)\alpha_k(-1)z^0 \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \beta_i(1)^2 \alpha_j(-1)\alpha_k(-1)z^{-1} \right) \mathbb{1} \otimes e^{\beta_{3-i}} \right) \\ &= (-1)^i \text{Res}_{z=0} \left( \left( I + h.o.t. \right) \left( \alpha_j(-1)\alpha_k(-1)z + (\beta_i, \alpha_j)\alpha_k(-1) + (\beta_i, \alpha_j)(\beta_i, \alpha_k)z^{-1} \right) \right. \\ &\quad \left. \cdot \mathbb{1} \otimes e^{\beta_{3-i}} \right) \end{aligned}$$



$$= (-1)^i (\beta_i, \alpha_j) (\beta_i, \alpha_k) \mathbb{1} \otimes e^{\beta_{3-i}},$$

which when  $j = k$  gives us  $F_i \cdot \alpha_j (-1)^2 \mathbb{1} \otimes e^{\beta_1 + \beta_2} = (-1)^i (\beta_i, \alpha_j)^2 \mathbb{1} \otimes e^{\beta_{3-i}}$ .

Let  $\overline{\mathbf{x}} = a\overline{\mathbf{p}}_1 + b\overline{\mathbf{p}}_2 + c\overline{\mathbf{p}}_3 \in \text{Ker}(F_1) \cap \text{Ker}(F_2)$  where  $a, b, c \in \mathbb{C}$ . Then

$$\begin{aligned} \overline{\mathbf{0}} &= F_i \cdot \overline{\mathbf{x}} = aF_i \cdot \left( \alpha_2(-2) + \alpha_2(-1)^2 \right) + bF_i \cdot \left( -\alpha_1(-1)^2 + \alpha_3(-1)^2 \right) \\ &\quad + cF_i \cdot \left( \alpha_1(-1)\alpha_3(-1) \right) \otimes \mathbb{1} e^{-\rho} \\ &= (-1)^i \left( a((\beta_i, \alpha_2) + (\beta_i, \alpha_2)^2) + b(-(\beta_i, \alpha_1)^2 + (\beta_i, \alpha_3)^2) \right. \\ &\quad \left. + c((\beta_i, \alpha_1)(\beta_i, \alpha_3)) \mathbb{1} \otimes e^{-\rho} \right) \\ &= \begin{cases} (-6a + 3b - 2c) \mathbb{1} \otimes e^{-\rho} & \text{if } i = 1, \\ (6a - 3b + 2c) \mathbb{1} \otimes e^{-\rho} & \text{if } i = 2. \end{cases} \end{aligned}$$

Thus we have  $\text{Low}_{\overline{P}_1}(-\rho) = \text{Span}(\overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2)$ , where

$$\overline{\mathbf{x}}_1 = \overline{\mathbf{p}}_1 + 2\overline{\mathbf{p}}_2$$

$$\overline{\mathbf{x}}_2 = -\overline{\mathbf{p}}_1 + 3\overline{\mathbf{p}}_3$$

□

Now we show that  $v_{-\rho}$  is represented in  $\text{Low}_{\overline{P}_1}(-\rho)$ .

**Proposition 6.5.**  $\pi_{\mathcal{F}}(v_{-\rho}) = \overline{\mathbf{x}}_2$ , where  $v_{-\rho} = -3e_{12123} + [E_2, 2E_1]$  is from Theorem 4.5.

*Proof.* Since  $\pi_{\mathcal{F}}$  is a Lie algebra representation, we have that  $\pi_{\mathcal{F}}(v_{-\rho}) = -3\pi_{\mathcal{F}}(e_{12123}) + 2\pi_{\mathcal{F}}([E_2, E_1])$ . We may use the restriction map  $\pi_{\mathcal{F}ib}$  for the second term, since the vector  $[E_2, E_1] \in \mathcal{F}ib$ . First we compute

$$\begin{aligned} \pi_{\mathcal{F}ib}([E_2, E_1]) &= [\pi_{\mathcal{F}ib}(E_2), \pi_{\mathcal{F}ib}(E_1)] = (\mathbb{1} \otimes e^{\beta_2})_0 (\mathbb{1} \otimes e^{\beta_1}) \\ &= \text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{\beta_2(-k)}{k} z^k \right) \exp \left( \sum_{k>0} \frac{\beta_2(k)}{-k} z^{-k} \right) e^{\beta_2} z^{\beta_2(0)} \varepsilon_{\beta_2} \right) (e^{\beta_1}) \\ &= \varepsilon(\beta_2, \beta_1) \text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{\beta_2(-k)}{k} z^k \right) (I + h.o.t.) z^{-3} \mathbb{1} \otimes e^{\beta_1 + \beta_2} \right) \\ &= -\text{Res}_{z=0} \left( (I + \beta_2(-1)z + \frac{1}{2}(\beta_2(-2) + \beta_2(-1)^2)z^2 + h.o.t.) z^{-3} \mathbb{1} \otimes e^{\beta_1 + \beta_2} \right) \\ &= -\frac{1}{2}(\beta_2(-2) + \beta_2(-1)^2) \mathbb{1} \otimes e^{\beta_1 + \beta_2}. \end{aligned}$$

Then, we compute  $\pi_{\mathcal{F}}(e_{12123})$  in steps. Using equation (B.1), we have

$$\begin{aligned} \pi_{\mathcal{F}}(e_{12123}) &= e_0^{\alpha_2} (\alpha_1(-1) \mathbb{1} \otimes e^{\alpha_1 + \alpha_2 + \alpha_3}) \\ &= \text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{\alpha_2(-k)}{k} z^k \right) \exp \left( \sum_{k>0} \frac{\alpha_2(k)}{-k} z^{-k} \right) e^{\alpha_2} z^{\alpha_2(0)} \varepsilon_{\alpha_2} \right) \alpha_1(-1) \mathbb{1} \otimes e^{\alpha_1 + \alpha_2 + \alpha_3} \end{aligned}$$

$$\begin{aligned}
 &= \varepsilon_{21}\varepsilon_{22}\varepsilon_{23}Res_{z=0}\left(\exp\left(\sum_{k>0}\frac{\alpha_2(-k)}{k}z^k\right)\left(I-\alpha_2(1)z^{-1}+h.o.t.\right)z^{-1}\right)\alpha_1(-1)\mathbb{1}\otimes e^{\alpha_1+2\alpha_2+\alpha_3} \\
 &= (1)(-1)(1)Res_{z=0}\left(\left(I+\alpha_2(-1)z+h.o.t.\right)\left(\alpha_1(-1)-(-2)z^{-1}\right)z^{-1}\right)\mathbb{1}\otimes e^{\alpha_1+2\alpha_2+\alpha_3} \\
 &= -\left(\alpha_1(-1)+2\alpha_2(-1)\right)\mathbb{1}\otimes e^{\alpha_1+2\alpha_2+\alpha_3},
 \end{aligned}$$

which gives us

$$\begin{aligned}
 \pi_{\mathcal{F}}(e_{12123}) &= e_0^{\alpha_1}\left(-\left(\alpha_1(-1)+2\alpha_2(-1)\right)\mathbb{1}\otimes e^{\alpha_1+2\alpha_2+\alpha_3}\right) \\
 &= -Res_{z=0}\left(\exp\left(\sum_{k>0}\frac{\alpha_1(-k)}{k}z^k\right)\exp\left(\sum_{k>0}\frac{\alpha_1(k)}{-k}z^{-k}\right)e^{\alpha_1}z^{\alpha_1(0)}\varepsilon_{\alpha_1}\right) \\
 &\quad \cdot \left(\alpha_1(-1)+2\alpha_2(-1)\right)\mathbb{1}\otimes e^{\alpha_1+2\alpha_2+\alpha_3} \\
 &= -\varepsilon_{11}(\varepsilon_{12})^2\varepsilon_{13}Res_{z=0}\left(\exp\left(\sum_{k>0}\frac{\alpha_1(-k)}{k}z^k\right)\left(I-\alpha_1(1)z^{-1}+h.o.t.\right)z^{-2}\right) \\
 &\quad \cdot \left((\alpha_1+2\alpha_2)(-1)\right)\mathbb{1}\otimes e^{2\alpha_1+2\alpha_2+\alpha_3} \\
 &= -(-1)(1)(1)Res_{z=0}\left(\left(I+\alpha_1(-1)z+\frac{1}{2}(\alpha_1(-2)+\alpha_1(-1)^2)z^2+h.o.t.\right)\right. \\
 &\quad \cdot \left.\left((\alpha_1+2\alpha_2)(-1)-(2+2(-2))z^{-1}\right)z^{-2}\right)\mathbb{1}\otimes e^{2\alpha_1+2\alpha_2+\alpha_3} \\
 &= \left(\alpha_1(-1)(\alpha_1+2\alpha_2)(-1)+2\left(\frac{1}{2}(\alpha_1(-2)+\alpha_1(-1)^2)\right)\right)\mathbb{1}\otimes e^{2\alpha_1+2\alpha_2+\alpha_3} \\
 &= \left(2\alpha_1(-1)^2+2\alpha_1(-1)\alpha_2(-1)+\alpha_1(-2)\right)\mathbb{1}\otimes e^{2\alpha_1+2\alpha_2+\alpha_3}. \\
 &= -\alpha_1(-1)\alpha_3(-1)\mathbb{1}\otimes e^{\beta_1+\beta_2}+q_1.
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
 \pi_{\mathcal{F}}(v_{-\rho}) &= -3\pi_{\mathcal{F}}(e_{12123})+2\pi_{\mathcal{F}}([E_2, E_1]) \\
 &= -3\left(-\alpha_1(-1)\alpha_3(-1)\mathbb{1}\otimes e^{\beta_1+\beta_2}\right)+2\left(-\frac{1}{2}(\beta_2(-2)+\beta_2(-1)^2)\mathbb{1}\otimes e^{\beta_1+\beta_2}\right) \\
 &= \left(3\alpha_1(-1)\alpha_3(-1)-\beta_2(-2)-\beta_2(-1)^2\right)\mathbb{1}\otimes e^{\beta_1+\beta_2} \\
 &= \left(3\alpha_1(-1)\alpha_3(-1)-\alpha_2(-2)-\alpha_2(-1)^2\right)\mathbb{1}\otimes e^{\beta_1+\beta_2} \\
 &= \overline{\mathfrak{x}}_2.
 \end{aligned}$$

□

Theorem 6.4 and Proposition 6.5 show that  $\overline{\mathfrak{x}}_1$  is a lowest-weight vector for  $\mathcal{F}ib$  in  $\overline{P}_1^{-\rho}$ , but  $\overline{\mathfrak{x}}_1 \notin \pi_{\mathcal{F}}(\mathcal{F})$ . Thus, the outer multiplicity data produced by this approach would only show how  $\overline{P}_1$ , not  $\mathcal{F}$ , decomposes with respect to  $\pi_{\mathcal{F}}(\mathcal{F}ib)$ . In this sense, the embedding of  $\mathcal{F}$  in  $\overline{P}_1$  does not help in analyzing how  $\mathcal{F}$  decomposes with respect to  $\mathcal{F}ib$ . In addition, it is also clear that this type of vertex algebra computation becomes more complex when examining  $\pi_{\mathcal{F}}(\mathcal{F}_{\mu}) \subset \overline{P}_1^{\mu}$  as  $|wt(\mu)|$  increases, rendering this approach less useful than anticipated.

However, Theorem 6.5 leads us to conjecture the existence of an algorithm based on the Schur polynomials ([B]) of vectors in  $\overline{P}_1$  which will allow one to determine which vectors are representations of vectors in  $\mathcal{F}$ , and/or which vectors are extremal with respect to  $\mathcal{F}ib$ . If such “recognition algorithms” are found, then a pattern might emerge within the set of extremal vectors for  $\mathcal{F}ib$  in  $\overline{P}_1$  which could shed light on the decomposition of  $\mathcal{F}$  with respect to  $\mathcal{F}ib$ . We therefore consider this approach still to be valuable, and the research is ongoing.

## Appendix A Kac-Peterson formulas

The method of Kac and Peterson (provided as exercise in [K2]) is a recursive algorithm for determining root multiplicities of the adjoint representation of any KM algebra  $\mathcal{L}$  with simple roots  $\beta_i, 1 \leq i \leq \ell$ . Define for any  $\beta \in Q_+$ ,

$$c_\beta = \sum_{d|\beta} \frac{1}{d} \text{Mult}\left(\frac{\beta}{d}\right).$$

As a simple example, we observe that if  $\beta$  is primitive (that is,  $\beta = n_1\beta_1 + n_2\beta_2$  where  $n_1$  and  $n_2$  are coprime), then  $c_\beta = \text{Mult}(\beta)$ , giving the starting values  $c_{\beta_i} = \text{Mult}(\beta_i) = 1$  for  $i = 1, 2$ . It can be shown that the  $c_\beta$ 's follow the recursive relation

$$(\beta|\beta - 2\rho)c_\beta = \sum_{\substack{\beta = \beta' + \beta'' \\ \beta', \beta'' \in Q_+}} (\beta'|\beta'')c_{\beta'}c_{\beta''}.$$

We further observe that  $c_\beta$  is defined in such a way that a Möbius inversion might be possible, which could give a similar expression for multiplicities in terms of  $c_\beta$ 's. First note that, for any  $\beta \in Q_+$ ,  $\beta = n\gamma$ , where  $n \geq 1$  and  $\gamma$  is primitive. Let

$$F_\gamma(m) = mc_{m\gamma} = \sum_{d|m} \frac{m}{d} \text{Mult}\left(\frac{m\gamma}{d}\right).$$

The above sum can be rewritten to range over the divisors  $d' = \frac{m}{d}$  :

$$F_\gamma(m) = \sum_{d'|m} d' \text{Mult}(d'\gamma).$$

If we define  $f_\gamma(d) = d \text{Mult}(d\gamma)$ , then we have the Möbius inversion

$$m \text{Mult}(m\gamma) = f_\gamma(m) = \sum_{d|m} \mu(d) F_\gamma\left(\frac{m}{d}\right) = \sum_{d|m} \mu(d) \frac{m}{d} c_{\frac{m\gamma}{d}},$$

which after dividing both sides by  $m$  gives

$$\text{Mult}(m\gamma) = \sum_{d|m} \mu(d) \frac{1}{d} c_{\frac{m\gamma}{d}}.$$

Thus we have a recursive method for finding the multiplicity of  $\beta$  in terms of the  $c_\gamma$  where  $\gamma|\beta$ :

$$\text{Mult}(\beta) = \text{Mult}(n\gamma) = \sum_{d|n} \mu(d) \frac{1}{d} c_{\frac{\beta}{d}}.$$

## Appendix B Supplementary results

Some of the proofs in Chapters 2 and 6 required lengthy computations which are either shown here in their entirety, or were facilitated through the use of the results and identities below.

### B.1 Multibracket theorem and identities used in Chapter 2

**Theorem B.1.** *Let  $e_i \in \mathcal{F}_{\alpha_i}$ ,  $f_i \in \mathcal{F}_{-\alpha_i}$ ,  $e_{j_n \dots j_1} \in \mathcal{F}_{\alpha}$ , and  $f_{j_n \dots j_1} \in \mathcal{F}_{-\alpha}$  where  $\alpha = \sum_{k=1}^n \alpha_{j_k}$  and  $n \geq 2$ . Then*

$$[e_i, f_{j_n \dots j_1}] = \delta_{ij_1} a_{j_1 j_2} f_{j_n \dots j_2} - \sum_{m=2}^n \delta_{ij_m} \left( \sum_{k=1}^{m-1} a_{j_m j_k} \right) f_{j_n \dots \hat{j}_m \dots j_1},$$

and

$$[f_i, e_{j_n \dots j_1}] = \delta_{ij_1} a_{j_1 j_2} e_{j_n \dots j_2} - \sum_{m=2}^n \delta_{ij_m} \left( \sum_{k=1}^{m-1} a_{j_m j_k} \right) e_{j_n \dots \hat{j}_m \dots j_1},$$

where  $x_{j_n \dots \hat{j}_m \dots j_1} = x_{j_n \dots j_{m+1} j_{m-1} \dots j_1}$  for  $x \in \{e, f\}$ .

*Proof.* We prove the first identity by induction on the length of the multibracket,  $n$ . Fix  $1 \leq i \leq 3$ . To show the base case is true let  $n = 2$ . Then we have

$$\begin{aligned} [e_i, f_{j_2 j_1}] &= [[e_i, f_{j_2}], f_{j_1}] + [f_{j_2}, [e_i, f_{j_1}]] \\ &= \delta_{ij_2} [h_{j_2}, f_{j_1}] + \delta_{ij_1} [f_{j_2}, h_{j_1}] \\ &= -\delta_{ij_2} a_{j_2 j_1} f_{j_1} + \delta_{ij_1} a_{j_1 j_2} f_{j_2}. \end{aligned}$$

Now assume that the statement is true for all  $2 \leq n \leq N$  for some large integer  $N$ . Then

$$\begin{aligned} [e_i, f_{j_{N+1} \dots j_1}] &= [e_i, [f_{j_{N+1}}, f_{j_N \dots j_1}]] \\ &= [[e_i, f_{j_{N+1}}], f_{j_N \dots j_1}] + [f_{j_{N+1}}, [e_i, f_{j_N \dots j_1}]]. \end{aligned}$$

We now invoke the induction hypothesis for the bracket inside the second term.

$$\begin{aligned} &= \delta_{ij_{N+1}} [h_{j_{N+1}}, f_{j_N \dots j_1}] \\ &\quad + [f_{j_{N+1}}, \delta_{ij_1} a_{j_1 j_2} f_{j_N \dots j_2} - \sum_{m=2}^N \delta_{ij_m} \left( \sum_{k=1}^{m-1} a_{j_m j_k} \right) f_{j_N \dots \hat{j}_m \dots j_1}] \\ &= -\delta_{ij_{N+1}} \left( \sum_{k=1}^N a_{j_{N+1} j_k} \right) f_{j_N \dots j_1} \end{aligned}$$

$$+ \delta_{ij_1} a_{j_1 j_2} f_{j_{N+1} \dots j_2} - \sum_{m=2}^N \delta_{j_m j_m} \left( \sum_{k=1}^{m-1} a_{ij_k} \right) f_{j_{N+1} \dots \hat{j}_m \dots j_1}.$$

Observe that the first term can be absorbed into the last term's outer sum as the case  $m = N + 1$  (since  $f_{j_N \dots j_1} = f_{\hat{j}_{N+1} j_N \dots j_1}$ ), giving us

$$= \delta_{ij_1} a_{j_1 j_2} f_{j_{N+1} \dots j_2} - \sum_{m=2}^{N+1} \delta_{j_m j_m} \left( \sum_{k=1}^{m-1} a_{ij_k} \right) f_{j_{N+1} \dots \hat{j}_m \dots j_1}.$$

The second identity follows from the first via the Cartan involution  $\nu$ .

□

The following identities were obtained using Theorem B.1, and are used in the proofs of Chapter 2.

$$[e_1, f_{1123}] = a_{32}(0) - \left( (a_{13} + a_{12})f_{123} + (a_{13} + a_{12} + a_{11})f_{123} \right) = 2f_{123},$$

$$[e_1, f_{123}] = a_{32}(0) - (a_{13} + a_{12})f_{23} = 2f_{23},$$

$$[f_1, e_{123}] = 2e_{23},$$

$$[e_2, f_{123}] = a_{32}(0) - (a_{23})f_{13} = f_{13} = 0,$$

$$[e_3, f_{123}] = (a_{32})f_{12} - (0) = -f_{12},$$

$$[e_2, f_{23}] = a_{32}(0) - (a_{23})f_3 = f_3,$$

$$[e_3, f_{23}] = a_{32}f_2 - (0) = -f_2,$$

$$[e_2, f_{12}] = a_{21}f_1 = -2f_1.$$

We also have the following immediate consequence of the theorem.

**Corollary B.2.** *Let  $1 \leq i \leq \ell$ , and let  $X$  be a multibracket  $f_{i_1 i_2 \dots i_n}$  (resp.  $e_{i_1 i_2 \dots i_n}$ ) such that  $i_j \neq i$  for all  $1 \leq j \leq n$ . Then  $[e_i, X] = 0$  (resp.  $[f_i, X] = 0$ ).*

In Chapter 6 the following Lemma aided in vertex operator computations.

**Lemma B.3.** *Let  $\beta, \gamma \in Q_{\mathcal{Fib}}$ . Then for  $k \geq 1$  and  $n > 0$ ,*

$$(i) \quad \beta(n)\gamma(-n)^k \mathbb{1} = nk(\beta, \gamma)\gamma(-n)^{k-1} \mathbb{1},$$

$$(ii) \quad \beta(n)^m \gamma(-n)^k \mathbb{1} = \frac{k!}{(k-m)!} n^m (\beta, \gamma)^m \gamma(-n)^{k-m} \mathbb{1}, \text{ for } k \geq m > 0.$$

*Proof.* (i) Induct on  $k$ . Base case:

$$\beta(n)\gamma(-n) \mathbb{1} = [\beta(n), \gamma(-n)] \mathbb{1} + \gamma(-n)\beta(n) \mathbb{1} = n(\beta, \gamma) \mathbb{1}.$$

Assume inductive hypothesis for all  $k \leq K$  for some  $K \in \mathbb{Z}_{>0}$ . Then

$$\begin{aligned}
 \beta(n)\gamma(-n)^{K+1}\mathbb{1} &= \left(\beta(n)\gamma(-n)\right)\gamma(-n)^K\mathbb{1} \\
 &= \left([\beta(n), \gamma(n)] + \gamma(-n)\beta(n)\right)\gamma^K(-n)\mathbb{1} \\
 &= n(\beta, \gamma)\gamma(-n)^K\mathbb{1} + \gamma(-n)\left(\beta(n)\gamma(-n)^K\mathbb{1}\right) \\
 &= n(\beta, \gamma)\gamma(-n)^K\mathbb{1} + \gamma(-n)\left(nK(\beta, \gamma)\gamma(-n)^{K-1}\mathbb{1}\right) \\
 &= n(K+1)(\beta, \gamma)\gamma(-n)^K\mathbb{1}.
 \end{aligned}$$

The induction hypothesis was applied in the fourth equality. Thus the statement is true for all  $k \in \mathbb{Z}$ .

(ii) Induct on  $m$  for fixed  $k \in \mathbb{Z}_{\geq 0}$ . Base case: When  $m = 1$  and  $k \geq 0$  we have part (i).

Assume that

$$\beta(n)^m\gamma(-n)^k\mathbb{1} = \frac{k!}{(k-m)!}n^m(\beta, \gamma)^m\gamma(-n)^{k-m}\mathbb{1}$$

is true for all  $m \leq M$  for some  $M \in \mathbb{Z}_{>0}$ . Then

$$\begin{aligned}
 \beta(n)^{M+1}\gamma(-n)^k\mathbb{1} &= \beta(n)\left(\beta(n)^M\gamma(-n)^k\mathbb{1}\right) \\
 &= \beta(n)\left(\frac{k!}{(k-M)!}n^M(\beta, \gamma)^M\gamma(-n)^{k-M}\mathbb{1}\right) \\
 &= \frac{k!}{(k-M)!}n^M(\beta, \gamma)^M\left(\beta(n)\gamma(-n)^{k-M}\mathbb{1}\right) \\
 &= \frac{k!}{(k-M)!}n^M(\beta, \gamma)^M\left((k-M)n(\beta, \gamma)\gamma(-n)^{k-M-1}\mathbb{1}\right) \\
 &= \frac{k!}{(k-(M+1))!}n^{M+1}(\beta, \gamma)^{M+1}\gamma(-n)^{k-(M+1)}\mathbb{1}.
 \end{aligned}$$

□

## B.2 The Lie algebra representation $\pi_{\mathcal{F}} : \mathcal{F} \rightarrow \overline{P}_1$

We now show that the representation  $\pi_{\mathcal{F}} : \mathcal{F} \rightarrow \overline{P}_1$  from Section 6.3 is a Lie algebra representation (i.e., the representations of the Serre generators satisfy the Serre relations in  $\mathcal{F}$  given in Definition 1.13). Note that since we are working with the representation of  $\mathcal{F}$  in  $V_{\mathcal{F}}$ , the 2-cocycle is  $\varepsilon_{ij}^{\mathcal{F}}$ , but we will suppress the superscript for brevity. We verify the first four Serre relations as follows:

$$\begin{aligned}
 [\pi_{\mathcal{F}}(h_i), \pi_{\mathcal{F}}(h_j)] &= (\alpha_i(-1)\mathbb{1} \otimes e^0)_0(\alpha_j(-1)\mathbb{1} \otimes e^0) \\
 &= \text{Res}_{z=0}(\alpha_i(z))\alpha_j(-1)\mathbb{1} \otimes e^0 = \text{Res}_{z=0}\left(\sum_{n \in \mathbb{Z}} \alpha_i(n)z^{-n-1}\right)\alpha_j(-1)\mathbb{1} \otimes e^0 \\
 &= \alpha_i(0)\alpha_j(-1)\mathbb{1} \otimes e^0 = (\alpha_i, 0)\alpha_j(-1)\mathbb{1} \otimes e^0 = 0,
 \end{aligned}$$

and

$$\begin{aligned} [\pi_{\mathcal{F}}(h_i), \pi_{\mathcal{F}}(e_j)] &= (\alpha_i(-1)\mathbb{1} \otimes e^0)_0(\mathbb{1} \otimes e^{\alpha_j}) = \text{Res}_{z=0}(\alpha_i(z))(\mathbb{1} \otimes e^{\alpha_j}) \\ &= \alpha_i(0)\mathbb{1} \otimes e^{\alpha_j} = (\alpha_i, \alpha_j)\mathbb{1} \otimes e^{\alpha_j} = a_{ji}\mathbb{1} \otimes e^{\alpha_j} = a_{ji}\pi_{\mathcal{F}}(e_j), \end{aligned}$$

and

$$\begin{aligned} [\pi_{\mathcal{F}}(h_i), \pi_{\mathcal{F}}(f_j)] &= (\alpha_i(-1)\mathbb{1} \otimes e^0)_0(-\mathbb{1} \otimes e^{-\alpha_j}) = -\text{Res}_{z=0}(\alpha_i(z))(\mathbb{1} \otimes e^{-\alpha_j}) \\ &= -\alpha_i(0)\mathbb{1} \otimes e^{-\alpha_j} = -(\alpha_i, -\alpha_j)\mathbb{1} \otimes e^{-\alpha_j} = a_{ji}\mathbb{1} \otimes e^{-\alpha_j} = -a_{ji}\pi_{\mathcal{F}}(f_j), \end{aligned}$$

and

$$\begin{aligned} [\pi_{\mathcal{F}}(e_i), \pi_{\mathcal{F}}(f_j)] &= -(\mathbb{1} \otimes e^{\alpha_i})_0(\mathbb{1} \otimes e^{-\alpha_j}) = -Y_0(\mathbb{1} \otimes e^{\alpha_i}, z)(\mathbb{1} \otimes e^{-\alpha_j}) \\ &= -\text{Res}_{z=0}\left(\exp\left(\sum_{k>0} \frac{\alpha_i(-k)}{k} z^k\right) \exp\left(\sum_{k>0} \frac{\alpha_i(k)}{-k} z^{-k}\right) e^{\alpha_i} z^{\alpha_i(0)} \varepsilon_{\alpha_i}\right)(\mathbb{1} \otimes e^{-\alpha_j}) \\ &= -\varepsilon(\alpha_i, -\alpha_j) \text{Res}_{z=0}\left(\exp\left(\sum_{k>0} \frac{\alpha_i(-k)}{k} z^k\right) \exp\left(\sum_{k>0} \frac{\alpha_i(k)}{-k} z^{-k}\right) z^{(\alpha_i, -\alpha_j)}\right)(\mathbb{1} \otimes e^{\alpha_i - \alpha_j}) \\ &= -\varepsilon_{ij}^{-1} \text{Res}_{z=0}\left(\exp\left(\sum_{k>0} \frac{\alpha_i(-k)}{k} z^k\right) (I + h.o.t.) z^{-a_{ji}}\right)(\mathbb{1} \otimes e^{\alpha_i - \alpha_j}) \\ &= -\varepsilon_{ij}^{-1} \text{Res}_{z=0}\left(I + \alpha_i(-1)z + h.o.t.\right) z^{-a_{ji}}(\mathbb{1} \otimes e^{\alpha_i - \alpha_j}) = -\delta_{ij} \varepsilon_{ij}^{-1} \alpha_i(-1) \mathbb{1} \otimes e^0 \\ &= -\delta_{ij} \varepsilon_{ij}^{-1} \pi_{\mathcal{F}}(h_i) = \delta_{ij} \pi_{\mathcal{F}}(h_i). \end{aligned}$$

Note in the fourth to last equality above if  $a_{ji} \leq 0$  then the residue is 0 since there would only be positive powers of  $z$  in the resulting series. If  $a_{ji} > 0$  then  $a_{ji} = 2$  and  $i = j$ , so the result is  $-\varepsilon_{ii}^{-1} \alpha_i(-1) \mathbb{1} \otimes e^0 = \alpha_i(-1) \mathbb{1} \otimes e^0$ , since we have chosen  $\varepsilon_{ii} = -1$  for all  $i = 1, 2, 3$ .

Lastly we need to check that the Serre relations  $(ad_{e_i})^{-a_{ij}+1}(e_j) = 0$  and  $(ad_{f_i})^{-a_{ij}+1}(f_j) = 0$  for all  $i, j = 1, 2, 3$  where  $i \neq j$  are correctly represented in  $\overline{P}_1$ .

The next page of calculations will show that  $\pi_{\mathcal{F}}((ad_{e_i})^{-a_{ij}+1}(e_j)) = 0$ . We start with  $\pi_{\mathcal{F}}([e_i, e_j])$  for general  $1 \leq i, j \leq 3$ , then substitute for each of the three cases  $a_{ij} = 0, -1$ , and  $-2$ :

$$\begin{aligned} [\pi_{\mathcal{F}}(e_i), \pi_{\mathcal{F}}(e_j)] &= [\mathbb{1} \otimes e^{\alpha_i}, \mathbb{1} \otimes e^{\alpha_j}] = (\mathbb{1} \otimes e^{\alpha_i})_0(\mathbb{1} \otimes e^{\alpha_j}) \\ &= \text{Res}_{z=0}\left(\exp\left(\sum_{k>0} \frac{\alpha_i(-k)}{k} z^k\right) \exp\left(\sum_{k>0} \frac{\alpha_i(k)}{-k} z^{-k}\right) e^{\alpha_i} z^{\alpha_i(0)} \varepsilon_{\alpha_i}\right)(\mathbb{1} \otimes e^{\alpha_j}) \\ &= \varepsilon_{ij} \text{Res}_{z=0}\left(\exp\left(\sum_{k>0} \frac{\alpha_i(-k)}{k} z^k\right) (I + h.o.t.) z^{(\alpha_i, \alpha_j)}\right)(\mathbb{1} \otimes e^{\alpha_i + \alpha_j}) \\ &= \varepsilon_{ij} \text{Res}_{z=0}\left(I + \alpha_i(-1)z + \frac{1}{2}(\alpha_i(-1)^2 + \alpha_i(-2))z^2 + h.o.t.\right) z^{a_{ji}}(\mathbb{1} \otimes e^{\alpha_i + \alpha_j}) \\ &= \begin{cases} 0 & \text{if } a_{ij} \geq 0, \\ \varepsilon_{ij} \mathbb{1} \otimes e^{\alpha_2 + \alpha_3} & \text{if } a_{ij} = -1, \\ \varepsilon_{ij} \alpha_i(-1) \mathbb{1} \otimes e^{\alpha_1 + \alpha_2} & \text{if } a_{ij} = -2. \end{cases} \end{aligned}$$



The first case above verifies the Serre relation for  $i = 1, j = 3$  (and similarly for  $i = 3, j = 1$ ), where  $a_{13} = 0$ . Continuing to check the other two cases, we first determine  $\pi_{\mathcal{F}}((ad_{e_2})^2(e_3))$ . We have

$$\begin{aligned} [\pi_{\mathcal{F}}(e_2), [\pi_{\mathcal{F}}(e_2), \pi_{\mathcal{F}}(e_3)]] &= (\mathbb{1} \otimes e^{\alpha_2})_0 (-\mathbb{1} \otimes e^{\alpha_2 + \alpha_3}) \\ &= -Res_{z=0} \left( \exp \left( \sum_{k>0} \frac{\alpha_2(-k)}{k} z^k \right) \exp \left( \sum_{k>0} \frac{\alpha_2(k)}{-k} z^{-k} \right) e^{\alpha_2} z^{(\alpha_2, \alpha_2 + \alpha_3)} \varepsilon_{\alpha_2} \right) (\mathbb{1} \otimes e^{\alpha_2 + \alpha_3}) \\ &= -\varepsilon_{22} \varepsilon_{23} Res_{z=0} \left( \exp \left( \sum_{k>0} \frac{\alpha_2(-k)}{k} z^k \right) (I + h.o.t.) z^1 \right) (\mathbb{1} \otimes e^{2\alpha_2 + \alpha_3}) = 0, \end{aligned}$$

since all of the negative powers of  $z$  from the annihilator expansion have coefficients which kill  $\mathbb{1}$ , leaving only positive powers of  $z$  in the final expansion. Thus the residue is 0, and for  $i = 2, j = 3$  (or for  $i = 3, j = 2$ ) where  $a_{ij} = -1$ , the Serre relation  $(ad_{e_2})^{-a_{23}+1}(e_3) = [e_2, [e_2, e_3]] = 0$  is correctly represented in  $\overline{P}_1$ . Lastly we compute  $\pi_{\mathcal{F}}((ad_{e_1})^3(e_2))$ ,

$$\begin{aligned} [\pi_{\mathcal{F}}(e_1), \pi_{\mathcal{F}}(e_{12})] &= (\mathbb{1} \otimes e^{\alpha_1})_0 (\alpha_1(-1) \mathbb{1} \otimes e^{\alpha_1 + \alpha_2}) \\ &= Res_{z=0} \left( \exp \left( \sum_{k>0} \frac{\alpha_1(-k)}{k} z^k \right) \exp \left( \sum_{k>0} \frac{\alpha_1(k)}{-k} z^{-k} \right) e^{\alpha_1} z^{(\alpha_1, \alpha_1 + \alpha_2)} \varepsilon_{\alpha_1} \right) (\alpha_1(-1) \mathbb{1} \otimes e^{\alpha_1 + \alpha_2}) \\ &= \varepsilon_{11} \varepsilon_{12} Res_{z=0} \left( \exp \left( \sum_{k>0} \frac{\alpha_1(-k)}{k} z^k \right) (I - \alpha_1(1) z^{-1} + h.o.t.) z^0 \right) (\alpha_1(-1) \mathbb{1} \otimes e^{2\alpha_1 + \alpha_2}) \\ &= \varepsilon_{11} \varepsilon_{12} Res_{z=0} (I + h.o.t.) (\alpha_1(-1) - 2z^{-1} + h.o.t.) \mathbb{1} \otimes e^{2\alpha_1 + \alpha_2} \\ &= -2\varepsilon_{11} \varepsilon_{12} \mathbb{1} \otimes e^{2\alpha_1 + \alpha_2}, \end{aligned}$$

then

$$\begin{aligned} [\pi_{\mathcal{F}}(e_1), \pi_{\mathcal{F}}(e_{112})] &= (\mathbb{1} \otimes e^{\alpha_1})_0 (-2\varepsilon_{11} \varepsilon_{12} \mathbb{1} \otimes e^{2\alpha_1 + \alpha_2}) \\ &= -\varepsilon_{11} \varepsilon_{12} Res_{z=0} \left( \exp \left( \sum_{k>0} \frac{\alpha_1(-k)}{k} z^k \right) \exp \left( \sum_{k>0} \frac{\alpha_1(k)}{-k} z^{-k} \right) e^{\alpha_1} z^{(\alpha_1, 2\alpha_1 + \alpha_2)} \varepsilon_{\alpha_1} \right) (2\mathbb{1} \otimes e^{2\alpha_1 + \alpha_2}) \\ &= -(\varepsilon_{11})^3 (\varepsilon_{12})^2 Res_{z=0} \left( \exp \left( \sum_{k>0} \frac{\alpha_i(-k)}{k} z^k \right) (I + h.o.t.) z^2 \right) (2\mathbb{1} \otimes e^{3\alpha_1 + \alpha_2}) = 0, \end{aligned}$$

since the expansion has no negative exponents, so the Serre relation  $(ad_{e_1})^{-a_{12}+1}(e_2) = [e_1, [e_1, [e_1, e_2]] = 0$  is correctly represented in  $\overline{P}_1$ .

Using a similar approach, we now show that  $\pi_{\mathcal{F}}((ad_{f_i})^{-a_{ij}+1}(f_j)) = 0$ , starting as we did above with  $\pi_{\mathcal{F}}([f_i, f_j])$  for general  $1 \leq i, j \leq 3$ , then substituting for each of the three cases  $a_{ij} = 0, -1$ , and  $-2$ :

$$\begin{aligned} [\pi_{\mathcal{F}}(f_i), \pi_{\mathcal{F}}(f_j)] &= (\mathbb{1} \otimes e^{-\alpha_i})_0 (\mathbb{1} \otimes e^{-\alpha_j}) \\ &= Res_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\alpha_i(-k)}{k} z^k \right) \exp \left( \sum_{k>0} \frac{-\alpha_i(k)}{-k} z^{-k} \right) e^{-\alpha_i} z^{-\alpha_i(0)} \varepsilon_{-\alpha_i} \right) (\mathbb{1} \otimes e^{-\alpha_j}) \\ &= \varepsilon(-\alpha_i, -\alpha_j) Res_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\alpha_i(-k)}{k} z^k \right) (I + h.o.t.) z^{(-\alpha_i, -\alpha_j)} \right) (\mathbb{1} \otimes e^{-\alpha_i - \alpha_j}) \end{aligned}$$

$$\begin{aligned}
 &= \varepsilon_{ij} \text{Res}_{z=0} \left( I - \alpha_i(-1)z + \frac{1}{2}(\alpha_i(-1)^2 - \alpha_i(-2))z^2 + h.o.t. \right) z^{\alpha_{ji}} \left( \mathbb{1} \otimes e^{-\alpha_i - \alpha_j} \right) \\
 &= \begin{cases} 0 & \text{if } a_{ij} \geq 0, \\ \varepsilon_{ij} \mathbb{1} \otimes e^{-\alpha_2 - \alpha_3} & \text{if } a_{ij} = -1, \\ -\varepsilon_{ij} \alpha_i(-1) \mathbb{1} \otimes e^{-\alpha_1 - \alpha_2} & \text{if } a_{ij} = -2. \end{cases}
 \end{aligned}$$

As before, the first case shows that the Serre relations  $(ad_{f_1})^{-a_{13}+1}(f_3) = [f_1, f_3] = 0$  and  $(ad_{f_3})^{-a_{31}+1}(f_1) = [f_3, f_1] = 0$  are correctly represented in  $\overline{P}_1$ . Next, we have

$$\begin{aligned}
 [\pi_{\mathcal{F}}(f_2), \pi_{\mathcal{F}}(f_{23})] &= (\mathbb{1} \otimes e^{-\alpha_2})_0 (-\mathbb{1} \otimes e^{-\alpha_2 - \alpha_3}) \\
 &= -\text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\alpha_2(-k)}{k} z^k \right) \exp \left( \sum_{k>0} \frac{-\alpha_2(k)}{-k} z^{-k} \right) e^{-\alpha_2} z^{(-\alpha_2, -\alpha_2 - \alpha_3)} \varepsilon_{-\alpha_2} \right) \\
 &\quad \cdot (\mathbb{1} \otimes e^{-\alpha_2 - \alpha_3}) \\
 &= -\varepsilon_{22} \varepsilon_{23} \text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\alpha_2(-k)}{k} z^k \right) (I + h.o.t.) z^1 \right) (\mathbb{1} \otimes e^{2\alpha_2 + \alpha_3}) = 0,
 \end{aligned}$$

since all of the negative powers of  $z$  from the annihilator expansion have coefficients which kill  $\mathbb{1}$ , leaving only positive powers of  $z$  in the final expansion. Thus the residue is 0, so the Serre relation  $(ad_{f_2})^{-a_{23}+1}(f_3) = [f_2, [f_2, f_3]] = 0$  is correctly represented in  $\overline{P}_1$ . Lastly we compute  $\pi_{\mathcal{F}}((ad_{f_1})^3(f_2))$ ,

$$\begin{aligned}
 [\pi_{\mathcal{F}}(f_1), \pi_{\mathcal{F}}(f_{12})] &= (\mathbb{1} \otimes e^{-\alpha_1})_0 (-\alpha_1(-1) \mathbb{1} \otimes e^{-\alpha_1 - \alpha_2}) \\
 &= -\text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\alpha_1(-k)}{k} z^k \right) \exp \left( \sum_{k>0} \frac{-\alpha_1(k)}{-k} z^{-k} \right) e^{-\alpha_1} z^{(-\alpha_1, -\alpha_1 - \alpha_2)} \varepsilon_{-\alpha_1} \right) \\
 &\quad \cdot (\alpha_1(-1) \mathbb{1} \otimes e^{-\alpha_1 - \alpha_2}) \\
 &= -\varepsilon_{11} \varepsilon_{12} \text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\alpha_1(-k)}{k} z^k \right) (I + \alpha_1(1)z^{-1} + h.o.t.) z^0 \right) (\alpha_1(-1) \mathbb{1} \otimes e^{-2\alpha_1 - \alpha_2}) \\
 &= -\varepsilon_{11} \varepsilon_{12} \text{Res}_{z=0} \left( (I + h.o.t.) (\alpha_1(-1) - 2z^{-1} + h.o.t.) \right) \mathbb{1} \otimes e^{-2\alpha_1 - \alpha_2} = 2\varepsilon_{11} \varepsilon_{12} \mathbb{1} \otimes e^{-2\alpha_1 - \alpha_2},
 \end{aligned}$$

then

$$\begin{aligned}
 [\pi_{\mathcal{F}}(f_1), \pi_{\mathcal{F}}(f_{112})] &= (\mathbb{1} \otimes e^{-\alpha_1})_0 (2\varepsilon_{11} \varepsilon_{12} \mathbb{1} \otimes e^{-2\alpha_1 - \alpha_2}) \\
 &= \varepsilon_{11} \varepsilon_{12} \text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\alpha_1(-k)}{k} z^k \right) \exp \left( \sum_{k>0} \frac{-\alpha_1(k)}{-k} z^{-k} \right) e^{-\alpha_1} z^{(-\alpha_1, -2\alpha_1 - \alpha_2)} \varepsilon_{-\alpha_1} \right) \\
 &\quad \cdot (2\mathbb{1} \otimes e^{-2\alpha_1 - \alpha_2}) \\
 &= (\varepsilon_{11})^3 (\varepsilon_{12})^2 \text{Res}_{z=0} \left( \exp \left( \sum_{k>0} \frac{-\alpha_1(-k)}{k} z^k \right) (I + h.o.t.) z^2 \right) (2\mathbb{1} \otimes e^{3\alpha_1 + \alpha_2}) = 0,
 \end{aligned}$$

since the expansion has no negative exponents, so  $(ad_{f_1})^{-a_{12}+1}(f_2) = [f_1, [f_1, [f_1, f_2]] = 0$  is correctly represented in  $\overline{P}_1$ .

### B.3 Proving $\pi_{\mathcal{F}}|_{\mathcal{F}ib} = \pi_{\mathcal{F}ib}$

Now we verify that for  $i = 1, 2$ ,  $\pi_{\mathcal{F}ib} = \pi_{\mathcal{F}}|_{\mathcal{F}ib}$ , that is,

$$\pi_{\mathcal{F}}(E_i) = \mathbb{1} \otimes e^{\beta_i}, \quad \pi_{\mathcal{F}}(F_i) = -\mathbb{1} \otimes e^{-\beta_i}, \quad \text{and} \quad \pi_{\mathcal{F}}(H_i) = \beta_i(-1)\mathbb{1} \otimes e^0,$$

where  $E_1 = \frac{1}{2}e_{1123}$ ,  $E_2 = e_2$ ,  $F_1 = -\frac{1}{2}f_{1123}$ ,  $F_2 = f_2$ ,  $H_1 = 2h_1 + h_2 + h_3$ , and  $H_2 = h_2$ .

First, we have

$$\begin{aligned} \pi_{\mathcal{F}}(H_1) &= \pi_{\mathcal{F}}(2h_1 + h_2 + h_3) = 2\alpha_1(-1)\mathbb{1} \otimes e^0 + \alpha_2(-1)\mathbb{1} \otimes e^0 + \alpha_3(-1)\mathbb{1} \otimes e^0 \\ &= (2\alpha_1 + \alpha_2 + \alpha_3)(-1)\mathbb{1} \otimes e^0 = \beta_1(-1)\mathbb{1} \otimes e^0 \end{aligned}$$

and

$$\pi_{\mathcal{F}}(H_2) = \pi_{\mathcal{F}}(h_2) = \alpha_2(-1)\mathbb{1} \otimes e^0 = \beta_2(-1)\mathbb{1} \otimes e^0.$$

Next, we have  $\pi_{\mathcal{F}}(E_1) = \pi_{\mathcal{F}}\left(\frac{1}{2}e_{1123}\right) = \frac{1}{2}[\pi_{\mathcal{F}}(e_1), \pi_{\mathcal{F}}(e_{123})]$ . Since it will prove useful in Proposition 6.5, we first compute  $\pi_{\mathcal{F}}(e_{123})$ , and we recall that  $\pi_{\mathcal{F}}(e_{23}) = \varepsilon_{23}(e^{\alpha_2+\alpha_3})$ .

$$\begin{aligned} \pi_{\mathcal{F}}(e_{123}) &= [\pi_{\mathcal{F}}(e_1), \pi_{\mathcal{F}}(e_{23})] = e_0^{\alpha_1}(\varepsilon_{23}e^{\alpha_2+\alpha_3}) \\ &= Res_{z=0}\left(\exp\left(\sum_{k>0}\frac{\alpha_1(-k)}{k}z^k\right)\exp\left(\sum_{k>0}\frac{\alpha_1(k)}{-k}z^{-k}\right)e^{\alpha_1}z^{(\alpha_1,\alpha_2+\alpha_3)}\varepsilon_{\alpha_1}\right)(\varepsilon_{23}\mathbb{1} \otimes e^{\alpha_2+\alpha_3}) \\ &= \varepsilon_{12}\varepsilon_{13}\varepsilon_{23}Res_{z=0}\left(\exp\left(\sum_{k>0}\frac{\alpha_1(-k)}{k}z^k\right)(I + h.o.t.)z^{-2}\right)(\mathbb{1} \otimes e^{\alpha_1+\alpha_2+\alpha_3}) \\ &= \varepsilon_{12}\varepsilon_{13}\varepsilon_{23}Res_{z=0}\left((I + \alpha_1(-1)z + h.o.t.)z^{-2}\right)(\mathbb{1} \otimes e^{\alpha_1+\alpha_2+\alpha_3}) \\ &= \varepsilon_{12}\varepsilon_{13}\varepsilon_{23}\alpha_1(-1)\mathbb{1} \otimes e^{\alpha_1+\alpha_2+\alpha_3} \end{aligned}$$

Since  $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{13} = 1$ , we have

$$\pi_{\mathcal{F}}(e_{123}) = \alpha_1(-1)\mathbb{1} \otimes e^{\alpha_1+\alpha_2+\alpha_3}. \quad (\text{B.1})$$

Thus we have

$$\begin{aligned} \pi_{\mathcal{F}}(E_1) &= \frac{1}{2}e_0^{\alpha_1}\left(\varepsilon_{12}\varepsilon_{13}\varepsilon_{23}\alpha_1(-1)\mathbb{1} \otimes e^{\alpha_1+\alpha_2+\alpha_3}\right) \\ &= \frac{1}{2}\varepsilon_{12}\varepsilon_{13}\varepsilon_{23}Res_{z=0}\left(\exp\left(\sum_{k>0}\frac{\alpha_1(-k)}{k}z^k\right)\exp\left(\sum_{k>0}\frac{\alpha_1(k)}{-k}z^{-k}\right)e^{\alpha_1}z^{(\alpha_1,\alpha_1+\alpha_2+\alpha_3)}\varepsilon_{\alpha_1}\right) \\ &\quad \cdot \alpha_1(-1)\mathbb{1} \otimes e^{\alpha_1+\alpha_2+\alpha_3} \\ &= \frac{1}{2}\varepsilon_{11}(\varepsilon_{12})^2(\varepsilon_{13})^2\varepsilon_{23}Res_{z=0}\left(\exp\left(\sum_{k>0}\frac{\alpha_1(-k)}{k}z^k\right)(I - \alpha_1(1)z^{-1} + h.o.t.)z^0\right) \\ &\quad \cdot \alpha_1(-1)\mathbb{1} \otimes e^{2\alpha_1+\alpha_2+\alpha_3} \\ &= \frac{1}{2}\varepsilon_{11}(\varepsilon_{12})^2(\varepsilon_{13})^2\varepsilon_{23}Res_{z=0}\left((I + h.o.t.)(I - 2z^{-1})\right)\mathbb{1} \otimes e^{2\alpha_1+\alpha_2+\alpha_3} \end{aligned}$$

$$= -\varepsilon_{11}\varepsilon_{23}\mathbb{1} \otimes e^{2\alpha_1+\alpha_2+\alpha_3}.$$

Since  $\varepsilon_{11} = -1$  and  $\varepsilon_{23} = 1$  we have

$$\pi_{\mathcal{F}}(e_{1123}) = \mathbb{1} \otimes e^{2\alpha_1+\alpha_2+\alpha_3} = \mathbb{1} \otimes e^{\beta_1} = \pi_{\mathcal{F}ib}(E_1),$$

and since  $E_2 = e_2$  and  $\beta_2 = \alpha_2$ ,

$$\pi_{\mathcal{F}}(E_2) = \mathbb{1} \otimes e^{\beta_2}.$$

Also,

$$\begin{aligned} \pi_{\mathcal{F}}(F_1) &= \pi_{\mathcal{F}}\left(-\frac{1}{2}f_{1123}\right) \\ &= -\frac{1}{2}[\pi_{\mathcal{F}}(f_1), [\pi_{\mathcal{F}}(f_1), \pi_{\mathcal{F}}(f_{23})]] \\ &= -\frac{1}{2}e_0^{-\alpha_1}\left(\text{Res}_{z=0}\left(\exp\left(\sum_{k>0}\frac{-\alpha_1(-k)}{k}z^k\right)\exp\left(\sum_{k>0}\frac{-\alpha_1(k)}{-k}z^{-k}\right)e^{-\alpha_1}z^{(-\alpha_1, -\alpha_2-\alpha_3)}\varepsilon_{-\alpha_1}\right)\right. \\ &\quad \cdot (\varepsilon_{23}\mathbb{1} \otimes e^{-\alpha_2-\alpha_3}) \\ &= -\frac{1}{2}(\varepsilon_{23})(\varepsilon_{12}\varepsilon_{13})e_0^{-\alpha_1}\left(\text{Res}_{z=0}\left(\exp\left(\sum_{k>0}\frac{-\alpha_1(-k)}{k}z^k\right)(I + h.o.t.)z^{-2}\right)(\mathbb{1} \otimes e^{-\alpha_1-\alpha_2-\alpha_3})\right) \\ &= -\frac{1}{2}\varepsilon_{23}\varepsilon_{12}\varepsilon_{13}e_0^{-\alpha_1}\left(\text{Res}_{z=0}\left((I - \alpha_1(-1)z + h.o.t.)z^{-2}\right)(\mathbb{1} \otimes e^{-\alpha_1-\alpha_2-\alpha_3})\right) \\ &= -\frac{1}{2}\varepsilon_{23}\varepsilon_{12}\varepsilon_{13}e_0^{-\alpha_1}\left(-\alpha_1(-1)\mathbb{1} \otimes e^{-\alpha_1-\alpha_2-\alpha_3}\right) \\ &= \frac{1}{2}\varepsilon_{23}\varepsilon_{12}\varepsilon_{13}\text{Res}_{z=0}\left(\exp\left(\sum_{k>0}\frac{-\alpha_1(-k)}{k}z^k\right)\exp\left(\sum_{k>0}\frac{-\alpha_1(k)}{-k}z^{-k}\right)e^{-\alpha_1}z^{(-\alpha_1, -\alpha_1-\alpha_2-\alpha_3)}\varepsilon_{-\alpha_1}\right) \\ &\quad \cdot (\alpha_1(-1)\mathbb{1} \otimes e^{-\alpha_1-\alpha_2-\alpha_3}) \\ &= \frac{1}{2}(\varepsilon_{12})^2(\varepsilon_{13})^2\varepsilon_{11}\varepsilon_{23}\text{Res}_{z=0}\left(\exp\left(\sum_{k>0}\frac{\alpha_1(-k)}{k}z^k\right)(I + \alpha_1(1)z^{-1} + h.o.t.)z^0\right) \\ &\quad \cdot (\alpha_1(-1)\mathbb{1} \otimes e^{-2\alpha_1-\alpha_2-\alpha_3}) \\ &= \frac{1}{2}(\varepsilon_{12})^2(\varepsilon_{13})^2\varepsilon_{11}\varepsilon_{23}\text{Res}_{z=0}\left((I + h.o.t.)(I + 2z^{-1})\right)(\mathbb{1} \otimes e^{-2\alpha_1-\alpha_2-\alpha_3}) \\ &= \frac{1}{2}(\varepsilon_{12})^2(\varepsilon_{13})^2\varepsilon_{11}\varepsilon_{23}(2\mathbb{1} \otimes e^{-2\alpha_1-\alpha_2-\alpha_3}) = \varepsilon_{11}\varepsilon_{23}\mathbb{1} \otimes e^{-2\alpha_1-\alpha_2-\alpha_3}. \end{aligned}$$

Thus we have

$$\pi_{\mathcal{F}}(F_1) = -\mathbb{1} \otimes e^{-\beta_1}$$

and since  $F_2 = f_2$  and  $\beta_2 = \alpha_2$ ,

$$\pi_{\mathcal{F}}(F_2) = -\mathbb{1} \otimes e^{-\beta_2}.$$

## Appendix C   Dimension data for weight spaces in irreducible *Fib*-modules

Chapter 5 details our algorithm for determining the inner multiplicities of non-standard modules on levels  $\pm 1$  and  $\pm 2$ . For explanation of the notation and organization of the tables, we refer the reader to Remark 5.4 and the bulleted text on page 53.

Tables C1 and C2 show data for the non-standard modules on levels 1 and 2, respectively.

Tables C3 and C4 show data for the adjoint representation and the  $-\rho$ -module on level 0. Though not referenced within the text, these data are provided to the reader as a demonstration of the claim in Chapter 5 that this algorithmic approach can be used to determine inner multiplicities of any irreducible module, including non-standard ones.

Table C1: Determination of bases of multibrackets in weight spaces  $V_\mu^{\Lambda_1}$  where  $\mu = n_1\beta_1 + n_2\beta_2 + \Lambda_1$ .  
See page 53 for explanation of notation.

$n_1, n_2$	$X_\mu$	$F_1 X_\mu$	$F_2 X_\mu$	$\mu(H_1)$	$\mu(H_2)$
0,0	$[\ ]$	0	N/A	-2	2
1,0	$[1]$	$(2)_{0,0}$	0	0	-1
2,0	$[11]$	$(2)_{1,0}$	0	2	-4
1,1	$[21]$	0	$(1)_{1,0}$	-3	1
2,1	$[211]$	$(2)_{1,1}$	$(4)_{2,0}$	-1	-2
	$[121]$	$(3)_{1,1}$	$(1)_{2,0}$		
3,1	$[1211]$	$(1,2)_{2,1}$	0	1	-5
	$[1121]$	$(0,4)_{2,1}$			
2,2	$[2211]$	0	$(6,0)_{2,1}$	-4	0
	$[2121]$		$(1,2)_{2,1}$		
4,1	$\begin{bmatrix} 11211 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 11211 \\ 11211 \end{bmatrix}$	$(0,2)_{3,1}$	0	3	-8
	$[11121]$	$(0,3)_{3,1}$			
3,2	$[21211]$	$(1,2)_{2,2}$	$(5,0)_{3,1}$	-2	-3
	$[21121]$	$(0,4)_{2,2}$	$(0,5)_{3,1}$		
	$[12211]$	$(4,0)_{2,2}$	$(6,0)_{3,1}$		
	$[12121]$	$(0,4)_{2,2}$	$(1,2)_{3,1}$		
2,3	$[22211]$	0	$(6,0)_{2,2}$	-7	2
	$[22121]$		$(1,2)_{2,2}$		
4,2	$[211121]$	$(0,3,0,0)_{3,2}$	$(8)_{4,1}$	0	-6
	$[121211]$	$(2,0,1,2)_{3,2}$	$(5,0)_{3,1} = (\frac{2}{3}, \frac{5}{3} + 0)_{4,1} = (\frac{10}{3})_{4,1}$		
	$[121121]$	$(0,2,0,4)_{3,2}$	$(0,5)_{3,1} = (\frac{2}{3}, 0) + 5)_{4,1} = (5)_{4,1}$		
	$[112211]$	$(0,0,6,0)_{3,2}$	$(6,0)_{3,1} = (\frac{2}{3}, 6) + 0)_{4,1} = (4)_{4,1}$		
	$[112121]$	$(0,0,0,6)_{3,2}$	$(1,2)_{3,1} = (\frac{2}{3}, 1) + 2)_{4,1} = (\frac{8}{3})_{4,1}$		
3,3	$[221211]$	$(1,2)_{2,3}$	$(8,0,0,0)_{3,2}$	-5	-1
	$[221121]$	$(0,4)_{2,3}$	$(0,8,0,0)_{3,2}$		
	$[212211]$	$(4,0)_{2,3}$	$(6,0,3,0)_{3,2}$		
	$[212121]$	$(0,4)_{2,3}$	$(1,2,0,3)_{3,2}$		
	$[122211]$	$(7,0)_{2,3}$	$(0,0,6,0)_{3,2}$		
	$[122121]$	$(0,7)_{2,3}$	$(0,0,1,2)_{3,2}$		

$n_1, n_2$	$X_\mu$		$F_1 X_\mu$	$F_2 X_\mu$	$\mu(H_1)$	$\mu(H_2)$
2,4	$[222211]$	*	0	$(4, 0)_{2,3}$	-10	4
	$[222121] = \frac{1}{4}[222211]$			$(1, 0)_{2,3}$		
5,2	$[1211121] = (0, \frac{3}{2}, 0, -1)$	*	$(0, 0, 3, 0, 0)_{4,2}$	0	2	-9
	$[1121211]$					
	$[1121121]$					
	$[1112211]$					
	$[1112121]$					
4,3	$[2211121]$		$(0, 3, 0, 0, 0)_{3,3}$	$(14, 0, 0, 0, 0)_{4,2}$	-3	-4
	$[2121211]$		$(2, 0, 1, 2, 0, 0)_{3,3}$	$(\frac{10}{3}, 6, 0, 0, 0)_{4,2}$		
	$[2121121]$		$(0, 2, 0, 4, 0, 0)_{3,3}$	$(5, 0, 6, 0, 0)_{4,2}$		
	$[2112211]$		$(0, 0, 6, 0, 0, 0)_{3,3}$	$(4, 0, 0, 6, 0)_{4,2}$		
	$[2112121]$		$(0, 0, 0, 6, 0, 0)_{3,3}$	$(\frac{8}{3}, 0, 0, 6, 0)_{4,2}$		
	$[1221211]$		$(5, 0, 0, 1, 2, 0)_{3,3}$	$(0, 8, 0, 0, 0)_{4,2}$		
	$[1221121]$		$(0, 5, 0, 0, 4, 0)_{3,3}$	$(0, 0, 8, 0, 0)_{4,2}$		
	$[1212211]$		$(0, 0, 5, 0, 4, 0)_{3,3}$	$(0, 6, 0, 3, 0)_{4,2}$		
	$[1212121]$		$(0, 0, 0, 5, 0, 4)_{3,3}$	$(0, 1, 2, 0, 3)_{4,2}$		
	$[1122211]$		$(0, 0, 0, 12, 0)_{3,3}$	$(0, 0, 0, 6, 0)_{4,2}$		
3,4	$[1122121]$		$(0, 0, 0, 0, 12)_{3,3}$	$(0, 0, 0, 1, 2)_{4,2}$	-8	1
	$[2221211]$		$(1, 2)_{2,3}^2 = (1 + \frac{1}{4}(2))_{2,4} = (\frac{3}{2})_{2,4}$	$(9, 0, 0, 0, 0)_{3,3}$		
	$[2221121]$		$(0, 4)_{2,3}^2 = (0 + \frac{1}{4}(4))_{2,4} = (1)_{2,4}$	$(0, 9, 0, 0, 0)_{3,3}$		
	$[2212211]$		$(4, 0)_{2,3}^2 = (4 + \frac{1}{4}(0))_{2,4} = (4)_{2,4}$	$(6, 0, 4, 0, 0)_{3,3}$		
	$[2212121]$		$(0, 4)_{2,3}^2 = (0 + \frac{1}{4}(4))_{2,4} = (1)_{2,4}$	$(1, 2, 0, 4, 0, 0)_{3,3}$		
	$[2122211]$		$(7, 0)_{2,3}^2 = (7 + \frac{1}{4}(0))_{2,4} = (7)_{2,4}$	$(0, 0, 6, 0, 1, 0)_{3,3}$		
	$[2122121]$		$(0, 7)_{2,3}^2 = (0 + \frac{1}{4}(7))_{2,4} = (\frac{7}{4})_{2,4}$	$(0, 0, 1, 2, 0, 1)_{3,3}$		
	$[1222211] = (4, 0, -6, 0, 4, 0)$		$(10)_{2,4}$	$(0, 0, 0, 4, 0)_{3,3}$		
6,2	$[11121211]$	*	$(0, 0, 1, 2)_{5,2}$	0	4	-12
	$= (\frac{1}{4}, \frac{1}{2})$		$(0, 0, 0, 4)_{5,2}$			
	$[11121121]$		$(0, 0, 4, 0)_{5,2}$			
	$[1112121]$		$(0, 0, 0, 4)_{5,2}$			

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$n_1, n_2$	$X_\mu$	$F_1 X_\mu$	$F_2 X_\mu$	$\mu(H_1)$	$\mu(H_2)$
5,3	[21121211]	(0, 2, 0, 1, 2, 0, 0, 0, 0, 0, 0) <sub>4,3</sub>	(9, 0, 0, 0) <sub>5,2</sub>		
	[21121121]	(0, 0, 2, 0, 4, 0, 0, 0, 0, 0) <sub>4,3</sub>	(0, 9, 0, 0) <sub>5,2</sub>		
	[21112211]	(0, 0, 0, 6, 0, 0, 0, 0, 0, 0) <sub>4,3</sub>	(0, 0, 9, 0) <sub>5,2</sub>		
	[21112121]	(0, 0, 0, 0, 6, 0, 0, 0, 0, 0) <sub>4,3</sub>	(0, 0, 0, 9) <sub>5,2</sub>		
	[12211121]	(3, 0, 0, 0, 0, 3, 0, 0, 0, 0) <sub>4,3</sub>	(14, 0, 0, 0, 0) <sub>4,2</sub> = (0, 21, 0, -14) <sub>5,2</sub>		
	[12121211]	(0, 3, 0, 0, 0, 2, 0, 1, 2, 0, 0) <sub>4,3</sub>	( $\frac{10}{3}$ , 6, 0, 0) <sub>4,2</sub> = (6, 5, 0, - $\frac{10}{3}$ ) <sub>5,2</sub>		
	[12121121]	(0, 0, 3, 0, 0, 2, 0, 4, 0, 0) <sub>4,3</sub>	(5, 0, 6, 0, 0) <sub>4,2</sub> = (0, $\frac{27}{2}$ , 0, -5) <sub>5,2</sub>		
	[12112211]	(0, 0, 0, 3, 0, 0, 0, 6, 0, 0) <sub>4,3</sub>	(4, 0, 0, 6, 0) <sub>4,2</sub> = (0, 6, 6, -4) <sub>5,2</sub>	-1	-7
	[12112121]	(0, 0, 0, 0, 3, 0, 0, 0, 6, 0, 0) <sub>4,3</sub>	( $\frac{8}{3}$ , 0, 0, 0, 6) <sub>4,2</sub> = (0, 4, 0, $\frac{10}{3}$ ) <sub>5,2</sub>		
	[11221211]	(0, 0, 0, 0, 0, 8, 0, 0, 0, 1, 2) <sub>4,3</sub>	(0, 8, 0, 0, 0) <sub>4,2</sub> = (8, 0, 0, 0) <sub>5,2</sub>		
	[11221121]	(0, 0, 0, 0, 0, 0, 8, 0, 0, 0, 4) <sub>4,3</sub>	(0, 0, 8, 0, 0) <sub>4,2</sub> = (0, 8, 0, 0) <sub>5,2</sub>		
	[11212211]	(0, 0, 0, 0, 0, 0, 0, 8, 0, 4, 0) <sub>4,3</sub>	(0, 6, 0, 3, 0) <sub>4,2</sub> = (6, 0, 3, 0) <sub>5,2</sub>		
	[11212121]	(0, 0, 0, 0, 0, 0, 0, 0, 8, 0, 4) <sub>4,3</sub>	(0, 1, 2, 0, 3) <sub>4,2</sub> = (1, 2, 0, 3) <sub>5,2</sub>		
	[11122211]	(0, 0, 0, 0, 0, 0, 0, 0, 0, 15, 0) <sub>4,3</sub>	(0, 0, 0, 6, 0) <sub>4,2</sub> = (0, 0, 6, 0) <sub>5,2</sub>		
	[11122121]	(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 15) <sub>4,3</sub>	(0, 0, 0, 1, 2) <sub>4,2</sub> = (0, 0, 1, 2) <sub>5,2</sub>		
4,4	[22211121]	(0, 3, 0, 0, 0, 0) <sub>3,4</sub>	(18, 0, 0, 0, 0, 0, 0, 0, 0, 0) <sub>4,3</sub>		
	[22121211]	(2, 0, 1, 2, 0, 0) <sub>3,4</sub>	( $\frac{10}{3}$ , 10, 0, 0, 0, 0, 0, 0, 0, 0) <sub>4,3</sub>		
	[22121121]	(0, 2, 0, 4, 0, 0) <sub>3,4</sub>	(5, 0, 10, 0, 0, 0, 0, 0, 0, 0) <sub>4,3</sub>		
	[22112211]	(0, 0, 6, 0, 0, 0) <sub>3,4</sub>	(4, 0, 0, 10, 0, 0, 0, 0, 0, 0) <sub>4,3</sub>		
	[22112121]	(0, 0, 0, 6, 0, 0) <sub>3,4</sub>	( $\frac{8}{3}$ , 0, 0, 0, 10, 0, 0, 0, 0, 0) <sub>4,3</sub>		
	[21221211]	(5, 0, 0, 0, 1, 2) <sub>3,4</sub>	(0, 8, 0, 0, 4, 0, 0, 0, 0, 0) <sub>4,3</sub>		
	[21221121]	(0, 5, 0, 0, 4) <sub>3,4</sub>	(0, 0, 8, 0, 0, 4, 0, 0, 0, 0) <sub>4,3</sub>		
	[21212211]	(0, 0, 5, 0, 4, 0) <sub>3,4</sub>	(0, 6, 0, 3, 0, 0, 0, 4, 0, 0) <sub>4,3</sub>		
	[21212121]	(0, 0, 0, 5, 0, 4) <sub>3,4</sub>	(0, 1, 2, 0, 3, 0, 0, 0, 4, 0, 0) <sub>4,3</sub>		
	[21122211]	(0, 0, 0, 0, 12, 0) <sub>3,4</sub>	(0, 0, 0, 6, 0, 0, 0, 0, 4, 0) <sub>4,3</sub>	-6	-2
	[21122121]	(0, 0, 0, 0, 0, 12) <sub>3,4</sub>	(0, 0, 0, 1, 2, 0, 0, 0, 0, 4) <sub>4,3</sub>		
	[12221211]	(8, 0, 0, 0, 0, $\frac{3}{2}$ ) <sub>3,4</sub> = (14, 0, -9, 0, 6, 0) <sub>3,4</sub>	(0, 0, 0, 0, 9, 0, 0, 0, 0, 0) <sub>4,3</sub>		
	[12221121]	(0, 8, 0, 0, 0, 0, 1) <sub>3,4</sub> = (4, 8, -6, 0, 4, 0) <sub>3,4</sub>	(0, 0, 0, 0, 0, 9, 0, 0, 0, 0) <sub>4,3</sub>		
	[12212211]	(0, 0, 8, 0, 0, 0, 4) <sub>3,4</sub> = (16, 0, -16, 0, 16, 0) <sub>3,4</sub>	(0, 0, 0, 0, 0, 6, 0, 4, 0, 0) <sub>4,3</sub>		
	[12212121]	(0, 0, 0, 8, 0, 0, 1) <sub>3,4</sub> = (1, 0, -6, 8, 1, 0) <sub>3,4</sub>	(0, 0, 0, 0, 1, 2, 0, 4, 0, 0) <sub>4,3</sub>		
	[12122211]	(0, 0, 0, 0, 8, 0, 7) <sub>3,4</sub> = (28, 0, -42, 0, 36, 0) <sub>3,4</sub>	(0, 0, 0, 0, 0, 0, 6, 0, 1, 0) <sub>4,3</sub>		
	[12122121]	(0, 0, 0, 0, 0, 8, $\frac{7}{3}$ ) <sub>3,4</sub> = (7, 0, - $\frac{21}{2}$ , 0, 7, 8) <sub>3,4</sub>	(0, 0, 0, 0, 0, 0, 1, 2, 0, 1) <sub>4,3</sub>		

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$n_1, n_2$	$X_\mu$	$F_1 X_\mu$	$F_2 X_\mu$	$\mu(H_1)$	$\mu(H_2)$
3,5	$[22221211]$	0	$(8, 0, 0, 0, 0)_{3,4}$	-11	3
	$[22221121]$		$(0, 8, 0, 0, 0)_{3,4}$		
	$[22212211]$		$(6, 0, 3, 0, 0)_{3,4}$		
	$[22212121]$		$(1, 2, 0, 3, 0)_{3,4}$		
	$[22122211]$ $= (-\frac{3}{2}, 0, 2, 0)$		$(0, 0, 6, 0, 0)_{3,4}$		
3,6	$[22122121]$ $= (-\frac{1}{3}, -\frac{1}{6}, \frac{1}{3}, \frac{2}{3})$	0	$(0, 0, 1, 2, 0, 0)_{3,4}$	-14	5
	$[22222121]$		$(8, 0, 0, 0)_{3,5}$		
	$[22221121]$		$(0, 8, 0, 0)_{3,5}$		
	$[22221221]$ $= (-\frac{4}{3}, \frac{2}{3})$		$(6, 0, 0, 0)_{3,5}$		
	$[222212121]$ $= (0, -4)$		$(1, 2, 0, 0)_{3,5}$		
6,3	$[211112211]$ $= (0, 0, 4, 0, 0, 0, -6, 0, 0, 4, 0, -1, 0)$	$(0, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0)_{5,3}$	$(12, 0)_{6,2}$	1	-10
	$[211112121]$ $= (0, 0, 0, 4, 0, 0, 0, -6, 0, 0, 4, 0, -1)$	$(0, 0, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0)_{5,3}$	$(0, 12)_{6,2}$		
	$[121121211]$	$(1, 0, 0, 0, 2, 0, 1, 2, 0, 0, 0, 0, 0)_{5,3}$	$(9, 0, 0)_{5,2} = (\frac{9}{4}, \frac{9}{2})_{6,2}$		
	$[121121121]$	$(0, 1, 0, 0, 0, 2, 0, 4, 0, 0, 0, 0, 0)_{5,3}$	$(0, 9, 0)_{5,2} = (0, 9)_{6,2}$		
	$[121112211]$	$(0, 0, 1, 0, 0, 0, 6, 0, 0, 0, 0, 0, 0)_{5,3}$	$(0, 0, 9, 0)_{5,2} = (9, 0)_{6,2}$		
	$[121112121]$	$(0, 0, 0, 1, 0, 0, 0, 6, 0, 0, 0, 0, 0)_{5,3}$	$(0, 0, 0, 9)_{5,2} = (0, 9)_{6,2}$		
	$[112211121]$	$(0, 0, 0, 0, 4, 0, 0, 0, 0, 3, 0, 0, 0)_{5,3}$	$(0, 21, 0, -14)_{5,2} = (0, 7)_{6,2}$		
	$[112121211]$	$(0, 0, 0, 0, 4, 0, 0, 0, 2, 0, 1, 2, 0, 0)_{5,3}$	$(6, 5, 0, -\frac{10}{3})_{5,2} = (\frac{3}{2}, \frac{14}{3})_{6,2}$		
	$[112121121]$	$(0, 0, 0, 0, 0, 4, 0, 0, 2, 0, 4, 0, 0)_{5,3}$	$(0, \frac{27}{2}, 0, -5)_{5,2} = (0, \frac{17}{2})_{6,2}$		
	$[112112211]$	$(0, 0, 0, 0, 0, 4, 0, 0, 0, 6, 0, 0, 0)_{5,3}$	$(0, 6, -4)_{5,2} = (6, 2)_{6,2}$		
	$[112112121]$	$(0, 0, 0, 0, 0, 0, 4, 0, 0, 0, 6, 0, 0)_{5,3}$	$(0, 4, 0, \frac{10}{3})_{5,2} = (0, \frac{22}{3})_{6,2}$		
	$[1121212121]$	$(0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 0, 6, 0, 0)_{5,3}$	$(8, 0, 0, 0)_{5,2} = (2, 4)_{6,2}$		
	$[111221211]$	$(0, 0, 0, 0, 0, 0, 0, 0, 9, 0, 0, 0, 1, 2)_{5,3}$	$(0, 8, 0, 0)_{5,2} = (0, 8)_{6,2}$		
	$[111221121]$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 9, 0, 0, 0, 4)_{5,3}$	$(6, 0, 3, 0)_{5,2} = (\frac{9}{2}, 3)_{6,2}$		
	$[111212121]$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 9, 0, 4)_{5,3}$	$(1, 2, 0, 3)_{5,2} = (\frac{1}{4}, \frac{11}{2})_{6,2}$		
	$[111122211]$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 16, 0)_{5,3}$	$(0, 0, 6, 0)_{5,2} = (6, 0)_{6,2}$		
	$[111122121]$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 16)_{5,3}$	$(0, 0, 1, 2)_{5,2} = (1, 2)_{6,2}$		

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$n_1, n_2$	$X_\mu$	$F_1 X_\mu$	$F_2 X_\mu$	$\mu(H_1)$	$\mu(H_2)$
5,4 (continued)	[21221121]	(0,0,0,0,0,8,0,0,4,0,0,0,0,0,0) <sub>4,4</sub>	(0,8,0,0,0,0,0,0,0,0,0,0,0,0) <sub>5,3</sub>		
	[211212211]	(0,0,0,0,0,0,8,0,4,0,0,0,0,0,0) <sub>4,4</sub>	(6,0,3,0,0,0,0,0,0,0,0,0,0,0) <sub>5,3</sub>		
	[211212121]	(0,0,0,0,0,0,0,8,0,4,0,0,0,0,0) <sub>4,4</sub>	(1,2,0,3,0,0,0,0,0,0,0,0,0,0) <sub>5,3</sub>		
	[211122211]	(0,0,0,0,0,0,0,0,15,0,0,0,0,0,0) <sub>4,4</sub>	(0,0,6,0,0,0,0,0,0,0,0,0,0,0,0) <sub>5,3</sub>		
	[211122121]	(0,0,0,0,0,0,0,0,0,15,0,0,0,0,0) <sub>4,4</sub>	(0,0,1,2,0,0,0,0,0,0,0,0,0,0,0) <sub>5,3</sub>		
	[122211121]	(6,0,0,0,0,0,0,0,0,0,0,3,0,0,0) <sub>4,4</sub>	(0,0,0,0,18,0,0,0,0,0,0,0,0,0) <sub>5,3</sub>		
	[122121211]	(0,6,0,0,0,0,0,0,0,0,0,2,0,1,2,0,0) <sub>4,4</sub>	(0,0,0,0,0, $\frac{10}{3}$ ,10,0,0,0,0,0,0,0,0) <sub>5,3</sub>		
	[122121121]	(0,0,6,0,0,0,0,0,0,0,0,0,2,0,4,0,0) <sub>4,4</sub>	(0,0,0,0,5,0,10,0,0,0,0,0,0,0,0) <sub>5,3</sub>		
	[122112211]	(0,0,0,6,0,0,0,0,0,0,0,0,0,6,0,0) <sub>4,4</sub>	(0,0,0,0,4,0,0,10,0,0,0,0,0,0,0) <sub>5,3</sub>		
	[122112121]	(0,0,0,0,6,0,0,0,0,0,0,0,0,6,0,0) <sub>4,4</sub>	(0,0,0,0, $\frac{8}{3}$ ,0,0,0,10,0,0,0,0,0,0) <sub>5,3</sub>		
	[121222121]	(0,0,0,0,0,6,0,0,0,0,0,0,5,0,0,0,1,2) <sub>4,4</sub>	(0,0,0,0,0,8,0,0,0,4,0,0,0,0,0,0) <sub>5,3</sub>	-4	-5
	[121221121]	(0,0,0,0,0,0,6,0,0,0,0,0,5,0,0,4) <sub>4,4</sub>	(0,0,0,0,0,0,8,0,0,0,4,0,0,0,0) <sub>5,3</sub>		
	[121212211]	(0,0,0,0,0,0,0,6,0,0,0,0,0,5,0,4,0) <sub>4,4</sub>	(0,0,0,0,0,6,0,3,0,0,0,4,0,0,0) <sub>5,3</sub>		
	[121212121]	(0,0,0,0,0,0,0,0,6,0,0,0,0,5,0,4,0) <sub>4,4</sub>	(0,0,0,0,0,1,2,0,3,0,0,0,4,0,0) <sub>5,3</sub>		
	[121122211]	(0,0,0,0,0,0,0,0,0,6,0,0,0,0,0,12,0) <sub>4,4</sub>	(0,0,0,0,0,0,0,6,0,0,0,0,0,0,4,0) <sub>5,3</sub>		
	[121122121]	(0,0,0,0,0,0,0,0,0,0,6,0,0,0,0,0,0,12) <sub>4,4</sub>	(0,0,0,0,0,0,0,1,2,0,0,0,0,0,0,4) <sub>5,3</sub>		
	[112222121]	(0,0,0,0,0,0,0,0,0,0,0,20,0,-9,0,6,0) <sub>4,4</sub>	(0,0,0,0,0,0,0,0,0,0,0,9,0,0,0,0,0) <sub>5,3</sub>		
	[112221121]	(0,0,0,0,0,0,0,0,0,0,0,4,14,-6,0,4,0) <sub>4,4</sub>	(0,0,0,0,0,0,0,0,0,0,0,0,9,0,0,0,0) <sub>5,3</sub>		
	[112212211]	(0,0,0,0,0,0,0,0,0,0,0,16,0,-10,0,16,0) <sub>4,4</sub>	(0,0,0,0,0,0,0,0,0,0,0,6,0,4,0,0,0) <sub>5,3</sub>		
	[112212121]	(0,0,0,0,0,0,0,0,0,0,0,1,0,-6,14,1,0) <sub>4,4</sub>	(0,0,0,0,0,0,0,0,0,0,1,2,0,4,0,0) <sub>5,3</sub>		
	[112122211]	(0,0,0,0,0,0,0,0,0,0,0,28,0,-42,0,42,0) <sub>4,4</sub>	(0,0,0,0,0,0,0,0,0,0,0,0,6,0,1,0) <sub>5,3</sub>		
	[112122121]	(0,0,0,0,0,0,0,0,0,0,0,7,0,- $\frac{21}{2}$ ,0,7,14) <sub>4,4</sub>	(0,0,0,0,0,0,0,0,0,0,0,0,1,2,0,1) <sub>5,3</sub>		

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Table C2: Determination of bases of multibrackets in weight spaces  $V_\mu^{\Lambda_2}$  where  $\beta = n_1\beta_1 + n_2\beta_2 + \Lambda_2$ .  
See page 53 for explanation of notation.

$n_1, n_2$	$X_\mu$		$F_1 X_\mu$	$F_2 X_\mu$	$\mu(H_1)$	$\mu(H_2)$
0,1	[2]		0	$(1)_{0,0}$	-2	1
	[12]		$(2)_{0,1}$	0	0	-2
2,1	[112]		$(2)_{1,1}$	0	2	-5
	[212]		0	$(2)_{1,1}$	-3	0
2,2	[2112]		$(2)_{1,2}$	$(5)_{2,1}$	-1	-3
	[1212]		$(3)_{1,2}$	$(2)_{2,1}$		
1,3	[2212]		0	$(2)_{1,2}$	-6	2
	[12112]		$(1, 2)_{2,2}$	0	1	-6
3,2	[11212]		$(0, 4)_{2,2}$			
	[22112]		$(2)_{1,3}$	$(8, 0)_{2,2}$	-4	-1
2,3	[21212]		$(3)_{1,3}$	$(2, 3)_{2,2}$		
	[12212]		$(6)_{1,3}$	$(0, 2)_{2,2}$		
4,2	$[112112] = \frac{2}{3}[111212]$	*	$(0, 2)_{3,2}$	0	3	-9
	[111212]		$(0, 3)_{3,2}$			
3,3	[212112]		$(1, 2, 0)_{2,3}$	$(6, 0)_{3,2}$		
	[211212]		$(0, 4, 0)_{2,3}$	$(0, 6)_{3,2}$	-2	-4
	[122112]		$(4, 0, 2)_{2,3}$	$(8, 0)_{3,2}$		
	[121212]		$(0, 4, 3)_{2,3}$	$(2, 3)_{3,2}$		
	[112212]		$(0, 0, 10)_{2,3}$	$(0, 2)_{3,2}$		
2,4	[222112]		0	$(9, 0, 0)_{2,3}$		
	[221212]			$(2, 4, 0)_{2,3}$	-7	1
	[212212]			$(0, 2, 1)_{2,3}$		
4,3	[2111212]		$(0, 3, 0, 0)_{3,3}$	$(9)_{4,2}$		
	[1212112]		$(2, 0, 1, 2, 0)_{3,3}$	$(4)_{4,2}$		
	[1211212]		$(0, 2, 0, 4, 0)_{3,3}$	$(6)_{4,2}$	0	-7
	[1122112]		$(0, 0, 6, 0, 2)_{3,3}$	$(\frac{16}{3})_{4,2}$		
	[1121212]		$(0, 0, 0, 6, 3)_{3,3}$	$(\frac{13}{3})_{4,2}$		
	[1112212]		$(0, 0, 0, 0, 12)_{3,3}$	$(2)_{4,2}$		

$n_1, n_2$	$X_\mu$		$F_1 X_\mu$	$F_2 X_\mu$	$\mu(H_1)$	$\mu(H_2)$
3, 4	[2212112]		$(1, 2, 0)_{2,4}$	$(10, 0, 0, 0)_{3,3}$	-5	-2
	[2211212]		$(0, 4, 0)_{2,4}$	$(0, 10, 0, 0)_{3,3}$		
	[2122112]		$(4, 0, 2)_{2,4}$	$(8, 0, 4, 0)_{3,3}$		
	[2121212]		$(0, 4, 3)_{2,4}$	$(2, 3, 0, 4)_{3,3}$		
	[2112212]		$(0, 0, 10)_{2,4}$	$(0, 2, 0, 4)_{3,3}$		
	[1222112]		$(7, 0, 0)_{2,4}$	$(0, 0, 9, 0)_{3,3}$		
2, 5	[1221212]		$(0, 7, 0)_{2,4}$	$(0, 0, 2, 4)_{3,3}$	-10	3
	[1212212]		$(0, 0, 7)_{2,4}$	$(0, 0, 0, 2, 1)_{3,3}$		
	[2222112]		0	$(8, 0, 0)_{2,4}$		
	[2221212]	*		$(2, 3, 0)_{2,4}$		
5, 3	[2212212] = $(-\frac{1}{6}, \frac{2}{3})$		$(0, 0, 3, 0, 0, 0)_{4,3}$	0	2	-10
	[1211212] = $(0, \frac{3}{2}, 0, -1, \frac{1}{4})$					
	[1121212]					
	[11211212]	*				
	[11122112]					
	[11121212]					
4, 4	[1112212]		$(0, 0, 0, 6, 3)_{4,3}$	$(16, 0, 0, 0, 0)_{4,3}$	-3	-5
	[1112212]		$(0, 0, 0, 0, 12)_{4,3}$			
	[2211212]		$(0, 3, 0, 0, 0, 0)_{3,4}$			
	[2121212]		$(2, 0, 1, 2, 0, 0)_{3,4}$			
	[21211212]		$(0, 2, 0, 4, 0, 0)_{3,4}$			
	[21122112]		$(0, 0, 6, 0, 2, 0)_{3,4}$			
	[21121212]		$(0, 0, 6, 3, 0, 0)_{3,4}$			
	[2112212]		$(0, 0, 0, 12, 0, 0)_{3,4}$			
	[12212112]		$(5, 0, 0, 0, 1, 2, 0)_{3,4}$			
	[12211212]		$(0, 5, 0, 0, 0, 4, 0)_{3,4}$			
	[12122112]		$(0, 0, 5, 0, 4, 0, 2)_{3,4}$			
	[12121212]		$(0, 0, 0, 5, 0, 4, 3)_{3,4}$			
	[12112212]		$(0, 0, 0, 5, 0, 0, 10)_{3,4}$	$(0, 2, 0, 0, 4)_{4,3}$		
	[11222112]		$(0, 0, 0, 0, 12, 0, 0)_{3,4}$	$(0, 0, 9, 0, 0)_{4,3}$		
	[11221212]		$(0, 0, 0, 0, 0, 12, 0)_{3,4}$	$(0, 0, 2, 4, 0)_{4,3}$		
	[11212212]		$(0, 0, 0, 0, 0, 12, 0)_{3,4}$	$(0, 0, 2, 4, 0)_{4,3}$		
	[11212212]		$(0, 0, 0, 0, 0, 12)_{3,4}$	$(0, 0, 0, 2, 1)_{4,3}$		
	[11212212]		$(0, 0, 0, 0, 0, 12)_{3,4}$			

Continued from previous page

$n_1, n_2$	$X_\mu$	$F_1 X_\mu$	$F_2 X_\mu$	$\mu(H_1)$	$\mu(H_2)$
3,5	[22212112]	$(1, 2)_{2,5}$	$(12, 0, 0, 0, 0, 0)_{3,4}$		
	[22211212]	$(0, 4)_{2,5}$	$(0, 12, 0, 0, 0, 0)_{3,4}$		
	[22122112]	$(\frac{11}{3}, \frac{4}{3})_{2,5}$	$(8, 0, 6, 0, 0, 0)_{3,4}$		
	[22121212]	$(-\frac{1}{2}, 6)_{2,5}$	$(2, 3, 0, 6, 0, 0)_{3,4}$		
	[22112212]	$(-\frac{5}{3}, \frac{20}{3})_{2,5}$	$(0, 2, 0, 6, 0, 0)_{3,4}$		
	[21222112]	$(7, 0)_{2,5}$	$(0, 0, 9, 0, 2, 0)_{3,4}$	-8	0
	[21221212]	$(0, 7)_{2,5}$	$(0, 0, 2, 4, 0, 2)_{3,4}$		
	[21212212]	$(-\frac{7}{6}, \frac{14}{3})_{2,5}$	$(0, 0, 2, 1, 0, 2)_{3,4}$		
	[12222112] = (4, 0, -6, 0, 4, 0, 0, 0)	$(10, 0)_{2,5}$	$(0, 0, 0, 0, 8, 0, 0)_{3,4}$		
	[12221212]	$(0, 10)_{2,5}$	$(0, 0, 0, 0, 2, 3, 0)_{3,4}$		
2,6	[22222112]	0	$(5, 0)_{2,5}$	-13	5
	[22221212] = $\frac{2}{5}[22222112]$		$(2, 0)_{2,5}$		
6,3	[111212112] = $(\frac{1}{4}, \frac{1}{2}, -\frac{1}{5})$	$(0, 0, 1, 2, 0)_{5,3}$	0	4	-13
	[111211212] = $(0, 1, -\frac{3}{10})$	$(0, 0, 4, 0)_{5,3}$			
	[11112212]	$(0, 4, 0, 2)_{5,3}$			
	[111121212]	$(0, 0, 4, 3)_{5,3}$			
	[111112212]	$(0, 0, 0, 10)_{5,3}$			
5,4	[211212112]	$(0, 2, 0, 1, 2, 0, 0, 0, 0, 0, 0)_{4,4}$	$(10, 0, 0, 0, 0)_{5,3}$	-1	-8
	[211211212]	$(0, 0, 2, 0, 4, 0, 0, 0, 0, 0)_{4,4}$	$(0, 10, 0, 0, 0)_{5,3}$		
	[211122112]	$(0, 0, 0, 6, 0, 2, 0, 0, 0, 0)_{4,4}$	$(0, 0, 10, 0, 0)_{5,3}$		
	[211121212]	$(0, 0, 0, 0, 6, 3, 0, 0, 0, 0)_{4,4}$	$(0, 0, 0, 10, 0)_{5,3}$		
	[211112212]	$(0, 0, 0, 0, 12, 0, 0, 0, 0, 0)_{4,4}$	$(0, 0, 0, 0, 10)_{5,3}$		
	[122111212]	$(3, 0, 0, 0, 0, 3, 0, 0, 0, 0)_{4,4}$	$(0, 24, 0, -16, 4)_{5,3}$		
	[121212112]	$(0, 3, 0, 0, 0, 2, 0, 1, 2, 0, 0)_{4,4}$	$(7, 6, 0, -4, 1)_{5,3}$		
	[121211212]	$(0, 0, 3, 0, 0, 2, 0, 4, 0, 0, 0)_{4,4}$	$(0, 16, 0, -6, \frac{3}{2})_{5,3}$		
	[121122112]	$(0, 0, 0, 3, 0, 0, 0, 6, 0, 2, 0)_{4,4}$	$(0, 8, 7, -\frac{16}{3}, \frac{4}{3})_{5,3}$		
	[121121212]	$(0, 0, 0, 3, 0, 0, 0, 6, 3, 0, 0)_{4,4}$	$(0, \frac{13}{2}, 0, \frac{8}{3}, \frac{13}{12})_{5,3}$		
	[121112212]	$(0, 0, 0, 0, 3, 0, 0, 12, 0, 0)_{4,4}$	$(0, 3, 0, -2, \frac{15}{2})_{5,3}$		
	[112212112]	$(0, 0, 0, 0, 8, 0, 0, 0, 0, 1, 2, 0)_{4,4}$	$(10, 0, 0, 0, 0)_{5,3}$		
	Continued from previous page				

$n_1, n_2$	$X_\mu$		$F_1 X_\mu$	$F_2 X_\mu$	$\mu(H_1)$	$\mu(H_2)$
5,4 (continued)	[112211212]		(0,0,0,0,0,0,0,0,0,0,4,0) <sub>4,4</sub>	(0,10,0,0,0) <sub>5,3</sub>		
	[112122112]		(0,0,0,0,0,0,0,0,0,4,0,2) <sub>4,4</sub>	(8,0,4,0,0) <sub>5,3</sub>		
	[112121212]		(0,0,0,0,0,0,0,0,0,8,0,0,4,3) <sub>4,4</sub>	(2,3,0,4,0) <sub>5,3</sub>		
	[112112212]		(0,0,0,0,0,0,0,0,0,8,0,0,10) <sub>4,4</sub>	(0,2,0,0,4) <sub>5,3</sub>	-1	-8
	[111222112]		(0,0,0,0,0,0,0,0,0,0,15,0,4,4)	(0,0,9,0,0) <sub>5,3</sub>		
	[111221212]		(0,0,0,0,0,0,0,0,0,0,15,0) <sub>4,4</sub>	(0,0,2,4,0) <sub>5,3</sub>		
	[111221212] = (0,0,0,0, $\frac{1}{4}$ , 0,0,0,0, -1,0,0,0,0, $\frac{3}{2}$ ,0,0)	*	(0,0,0,0,0,0,0,0,0,0,0,15) <sub>4,4</sub>	(0,0,0,2,1) <sub>5,3</sub>		
	[11212212]		(0,3,0,0,0,0,0,0,0,0) <sub>3,5</sub>	(21,0,0,0,0,0,0,0,0,0,0,0) <sub>4,4</sub>		
	[22211212]		(2,0,1,2,0,0,0,0,0) <sub>3,5</sub>	(4,12,0,0,0,0,0,0,0,0,0,0) <sub>4,4</sub>		
	[22121212]		(0,2,0,4,0,0,0,0,0) <sub>3,5</sub>	(6,0,12,0,0,0,0,0,0,0,0,0) <sub>4,4</sub>		
4,5	[22122112]		(0,0,6,0,2,0,0,0,0) <sub>3,5</sub>	( $\frac{16}{3}$ ,0,0,12,0,0,0,0,0,0,0,0,0) <sub>4,4</sub>		
	[221121212]		(0,0,0,6,3,0,0,0,0) <sub>3,5</sub>	( $\frac{13}{3}$ ,0,0,0,12,0,0,0,0,0,0,0,0) <sub>4,4</sub>		
	[22112212]		(0,0,0,0,12,0,0,0,0) <sub>3,5</sub>	(2,0,0,0,0,0,12,0,0,0,0,0,0,0) <sub>4,4</sub>		
	[212212112]		(5,0,0,0,0,1,2,0,0) <sub>3,5</sub>	(0,10,0,0,0,0,5,0,0,0,0,0,0) <sub>4,4</sub>		
	[212211212]		(0,5,0,0,0,0,4,0,0) <sub>3,5</sub>	(0,0,10,0,0,0,0,5,0,0,0,0,0) <sub>4,4</sub>		
	[212122112]		(0,0,5,0,0,4,0,2,0) <sub>3,5</sub>	(0,8,0,4,0,0,0,0,5,0,0,0,0) <sub>4,4</sub>		
	[212121212]		(0,0,0,5,0,0,4,3,0) <sub>3,5</sub>	(0,2,3,0,4,0,0,0,5,0,0,0,0) <sub>4,4</sub>		
	[212112212]		(0,0,0,0,5,0,0,10,0) <sub>3,5</sub>	(0,0,2,0,0,4,0,0,0,5,0,0,0) <sub>4,4</sub>		
	[211222112]		(0,0,0,0,0,12,0,0,0) <sub>3,5</sub>	(0,0,0,9,0,0,0,0,0,0,5,0,0) <sub>4,4</sub>	-6	-3
	[211221212]		(0,0,0,0,0,0,12,0,0) <sub>3,5</sub>	(0,0,0,2,4,0,0,0,0,0,5,0) <sub>4,4</sub>		
	[211212212]		(0,0,0,0,0,0,0,12,0) <sub>3,5</sub>	(0,0,0,0,2,1,0,0,0,0,0,5) <sub>4,4</sub>		
	[122212112]		(12,0,-6,0,0,4,0,0,2) <sub>3,5</sub>	(0,0,0,0,0,12,0,0,0,0,0,0) <sub>4,4</sub>		
	[122211212]		(0,8,0,0,0,0,0,0,4) <sub>3,5</sub>	(0,0,0,0,0,0,12,0,0,0,0,0) <sub>4,4</sub>		
	[122122112]		( $\frac{44}{3}$ ,0,-14,0,0, $\frac{44}{3}$ ,0,0, $\frac{4}{3}$ ) <sub>3,5</sub>	(0,0,0,0,0,0,8,0,6,0,0,0,0) <sub>4,4</sub>		
	[122121212]		(-2,0,3,8,0,-2,0,0,6) <sub>3,5</sub>	(0,0,0,0,0,0,2,3,0,6,0,0,0) <sub>4,4</sub>		
	[122112212]		( $-\frac{20}{3}$ ,0,10,0,8,- $\frac{20}{3}$ ,0,0, $\frac{20}{3}$ ) <sub>3,5</sub>	(0,0,0,0,0,0,0,2,0,6,0,0,0) <sub>4,4</sub>		
	[121222112]		(28,0,-42,0,0,36,0,0,0) <sub>3,5</sub>	(0,0,0,0,0,0,0,9,0,0,2,0,0) <sub>4,4</sub>		
	[121221212]		(0,0,0,0,0,0,8,0,7) <sub>3,5</sub>	(0,0,0,0,0,0,0,2,4,0,0,2,0) <sub>4,4</sub>		
	[121212212]		( $-\frac{14}{3}$ ,0,7,0,0,- $\frac{14}{3}$ ,0,8, $\frac{14}{3}$ ) <sub>3,5</sub>	(0,0,0,0,0,0,0,0,2,1,0,0,2) <sub>4,4</sub>		
	[112221212]		(0,0,0,0,0,0,0,0,18) <sub>3,5</sub>	(0,0,0,0,0,0,0,0,0,2,3,0) <sub>4,4</sub>		
Continued from previous page						

$n_1, n_2$	$X_\mu$	$F_1 X_\mu$	$F_2 X_\mu$	$\mu(H_1)$	$\mu(H_2)$
3, 6	[222212112]	$(\frac{9}{5})_{2,6}$	(12, 0, 0, 0, 0, 0, 0, 0) <sub>3,5</sub>		
	[222211212]	$(\frac{9}{5})_{2,6}$	(0, 12, 0, 0, 0, 0, 0, 0) <sub>3,5</sub>		
	[222122112]	$(\frac{21}{5})_{2,6}$	(8, 0, 6, 0, 0, 0, 0, 0) <sub>3,5</sub>		
	[222121212]	$(-\frac{19}{10})_{2,6}$	(2, 3, 0, 6, 0, 0, 0, 0) <sub>3,5</sub>		
	[222112212]	(1) <sub>2,6</sub>	(0, 2, 0, 6, 0, 0, 0, 0) <sub>3,5</sub>	-11	2
	[221222112]	(7) <sub>2,6</sub>	(0, 0, 9, 0, 2, 0, 0, 0) <sub>3,5</sub>		
	[221221212]	$(\frac{14}{5})_{2,6}$	(0, 0, 2, 4, 0, 2, 0, 0) <sub>3,5</sub>		
	[221212212]	$(-\frac{7}{10})_{2,6}$	(0, 0, 2, 1, 0, 2, 0, 0) <sub>3,5</sub>		
	[212221212] = $(\frac{3}{5}, \frac{1}{4}, -2, -1, 0, 1, \frac{3}{5}, 0)$	(4) <sub>2,6</sub>	(0, 0, 0, 0, 2, 3, 0, 0) <sub>3,5</sub>		
	[122222112] = (15, 0, -20, 0, 0, 10, 0, 0)	(13) <sub>2,6</sub>	(20, 0, -30, 0, 0, 20, 0, 0) <sub>3,5</sub>		
	Continued from previous page				
		*			



Table C3: Determination of bases of multibrackets in the adjoint representation  $\mathcal{Fib}$ . Note:  $\beta = n_1\beta_1 + n_2\beta_2$ .  
See page 53 for explanation of notation.

$n_1, n_2$	$X_\beta$	$F_1X_\beta$	$F_2X_\beta$	$\beta(H_1)$	$\beta(H_2)$
$n_1, n_2$	$X_\beta$	$F_1X_\beta$	$F_2X_\beta$	$\beta(H_1)$	$\beta(H_2)$
1,1	[21]	$(-3)_{0,1}$	$(3)_{1,0}$	-2	-2
	[12] = -[21]	$(3)_{0,1}$	$(-3)_{1,0}$		
2,1	[121]	$(4)_{1,1}$	0	1	-4
1,2	[221]	0	$(4)_{1,1}$	-4	1
3,1	[1121]	$(3)_{2,1}$	0	3	-7
2,2	[2121]	$(4)_{1,2}$	$(4)_{2,1}$	-2	-2
	[1221] = [2121]	$(4)_{1,2}$	$(4)_{2,1}$		
1,3	[2221]	0	$(3)_{1,2}$	-7	3
3,2	[21121]	$(3)_{2,2}$	$(7)_{3,1}$	0	-5
	[12121]	$(6)_{2,2}$	$(4)_{3,1}$		
2,3	[22121]	$(4)_{2,2}$	$(6)_{3,1}$	-5	0
	[12221]	$(7)_{2,2}$	$(3)_{3,1}$		
4,2	[121121] = $\frac{1}{5}$ [112121]	$(0,3)_{3,2}$	0	2	-8
	[112121]	$(0,6)_{3,2}$			
3,3	[221121]	$(3,0)_{2,3}$	$(12,0)_{3,2}$	-3	-3
	[212121]	$(6,0)_{2,3}$	$(4,5)_{3,2}$		
	[1222121] = $(-\frac{1}{3}, 1, \frac{1}{3})$	$(5,4)_{2,3}$	$(0,6)_{3,2}$		
	[112221]	$(0,12)_{2,3}$	$(0,3)_{3,2}$		
2,4	[222121]	0	$(6,0)_{2,3}$	-8	2
5,2	[212221] = $\frac{1}{3}$ [222121]		$(3,0)_{2,3}$		
	[112121]	$(4)_{4,2}$	0	4	-11
4,3	[2112121]	$(0,6,0)_{3,3}$	$(8)_{4,2}$		
	[1221121]	$(2,3,1)_{3,3}$	$(6)_{4,2}$	-1	6
	[1212121]	$(-2,9,2)_{3,3}$	$(7)_{4,2}$		
	[1122221]	$(0,0,15)_{3,3}$	$(3)_{4,2}$		
3,4	[2221121]	$(3)_{2,4}$	$(15,0,0)_{3,3}$	6	-1
	[2212121]	$(6)_{2,4}$	$(4,8,0)_{3,3}$		
	[2112221]	$(6)_{2,4}$	$(0,3,3)_{3,3}$		
2,5	[1222121]	$(8)_{2,4}$	$(-2,6,2)_{3,3}$	-11	4
	[2222121]	0	$(4)_{2,4}$		

$n_1, n_2$	$X_\beta$		$F_1 X_\beta$	$F_2 X_\beta$	$\beta(H_1)$	$\beta(H_2)$
5,3	$[21112121] = (4, -\frac{4}{3}, -2, \frac{1}{3})$		$(4, 0, 0, 0)_{4,3}$	$(11)_{5,2}$	1	-9
	$[12112121]$		$(1, 0, 6, 0)_{4,3}$	$(8)_{5,2}$		
	$[11221121]$		$(0, 3, 3, 1)_{4,3}$	$(6)_{5,2}$		
	$[11212121]$		$(0, -2, 10, 2)_{4,3}$	$(7)_{5,2}$		
	$[11112221]$		$(0, 0, 0, 16)_{4,3}$	$(3)_{5,2}$		
4,4	$[22112121]$		$(0, 6, 0, 0)_{3,4}$	$(14, 0, 0, 0)_{4,3}$	-4	-4
	$[21221121]$		$(2, 3, 1, 0)_{3,4}$	$(6, 6, 0, 0)_{4,3}$		
	$[21212121]$		$(-2, 9, 2, 0)_{3,4}$	$(7, 0, 6, 0)_{4,3}$		
	$[21112221]$		$(0, 0, 15, 0)_{3,4}$	$(3, 0, 0, 6)_{4,3}$		
	$[12221121]$		$(6, 0, 0, 3)_{3,4}$	$(0, 15, 0, 0)_{4,3}$		
	$[12212121]$		$(0, 6, 0, 6)_{3,4}$	$(0, 4, 8, 0)_{4,3}$		
	$[12112221] = (0, \frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, -1, \frac{3}{2})$		$(0, 0, 6, 6)_{3,4}$	$(0, 0, 3, 3)_{4,3}$		
	$[11222121] = (1, 1, -3, \frac{1}{3}, -\frac{4}{3}, 3)$		$(0, 0, 0, 14)_{3,4}$	$(0, -2, 6, 2)_{4,3}$		
	$[2221121]$		$(3)_{2,5}$	$(16, 0, 0, 0)_{3,4}$		
3,5	$[22212121]$		$(6)_{2,5}$	$(4, 9, 0, 0)_{3,4}$	-9	1
	$[22112221]$		$(6)_{2,5}$	$(0, 3, 4, 0)_{3,4}$		
	$[21222121]$		$(8)_{2,5}$	$(-2, 6, 2, 1)_{3,4}$		
	$[12222121] = (1, -2, -2, 4)$		$(11)_{2,5}$	$(0, 0, 0, 4)_{3,4}$		
6,3	$[112112121] = (\frac{1}{5}, \frac{1}{2}, \frac{1}{10})$		$(0, 0, 6, 0)_{5,3}$	0	3	-12
	$[11122121]$		$(0, 2, 3, 1)_{5,3}$			
	$[111212121]$		$(0, -2, 9, 2)_{5,3}$			
	$[11112221]$		$(0, 0, 0, 15)_{5,3}$			
5,4	$[212112121]$		$(1, 0, 6, 0, 0)_{4,4}$	$(14, -\frac{32}{3}, -16, \frac{8}{3})_{5,3}$	-2	-7
	$[211221121]$		$(0, 3, 3, 1, 0)_{4,4}$	$(24, 1, -12, 2)_{5,3}$		
	$[211212121]$		$(0, -2, 10, 2, 0)_{4,4}$	$(28, -\frac{28}{3}, -5, \frac{7}{3})_{5,3}$		
	$[21112221]$		$(0, 0, 0, 16, 0)_{4,4}$	$(12, -4, -6, 10)_{5,3}$		
	$[122112121]$		$(4, 6, 0, 0, 0)_{4,4}$	$(14, 0, 0, 0)_{5,3}$		
	$[121221121]$		$(2, \frac{17}{2}, -\frac{3}{2}, \frac{1}{2}, -1, \frac{3}{2})_{4,4}$	$(6, 6, 0, 0)_{5,3}$		
	$[121212121]$		$(-2, 12, 1, 1, -2, 3)_{4,4}$	$(7, 0, 6, 0)_{5,3}$		
	$[12112221]$		$(0, \frac{45}{2}, -\frac{45}{2}, \frac{23}{2}, -15, \frac{45}{2})_{4,4}$	$(3, 0, 0, 6)_{5,3}$		
	$[112221121]$		$(9, 3, -9, 1, 0, 9)_{4,4}$	$(0, 15, 0, 0)_{5,3}$		
	$[112212121] = (0, 0, 0, -\frac{1}{3}, \frac{2}{3}, -\frac{7}{3}, 1, \frac{5}{9}, \frac{10}{9})$		$(6, 12, -18, 2, -8, 22)_{4,4}$	$(0, 4, 8, 0)_{5,3}$		

Continued from previous page

$n_1, n_2$	$X_\beta$	$F_1 X_\beta$	$F_2 X_\beta$	$\beta(H_1)$	$\beta(H_2)$
4,5	$[222112121] = (3, 0, 0, -2, 0, \frac{1}{2}, 0, 0, 0)$	$(0, 6, 0, 0)_{3,5}$	$(18, 0, 0, 0, 0)_{4,4}$		
	$[221221121]$	$(2, 3, 1, 0)_{3,5}$	$(6, 10, 0, 0, 0)_{4,4}$		
	$[221212121]$	$(-2, 9, 2, 0)_{3,5}$	$(7, 0, 10, 0, 0)_{4,4}$		
	$[221112221]$	$(0, 0, 15, 0)_{3,5}$	$(3, 0, 0, 10, 0, 0)_{4,4}$		
	$[212221121]$	$(6, 0, 0, 3)_{3,5}$	$(0, 15, 0, 0, 4, 0)_{4,4}$		
	$[212212121]$	$(0, 6, 0, 6)_{3,5}$	$(0, 4, 8, 0, 0, 4)_{4,4}$		
	$[122221121]$	$(12, -6, -6, 12)_{3,5}$	$(0, 0, 0, 0, 16, 0)_{4,4}$		
	$[122212121]$	$(6, -3, -12, 24)_{3,5}$	$(0, 0, 0, 0, 4, 9)_{4,4}$		
	$[122112221]$	$(6, -12, -3, 24)_{3,5}$	$(0, 6, -6, 2, -4, 9)_{4,4}$		
	$[121222121]$	$(8, -16, -16, 41)_{3,5}$	$(-1, 4, -6, \frac{4}{3}, \frac{-16}{3}, 12)_{4,4}$		
3,6	$[2122221121]$		$(15, 0, 0, 0)_{3,5}$		
	$[2122212121]$		$(4, 8, 0, 0)_{3,5}$		
	$[2122112221]$		$(0, 3, 3, 0)_{3,5}$		
	$[21212222121] = (-\frac{4}{15}, \frac{1}{2}, \frac{2}{3})$		$(-2, 6, 2, 0)_{3,5}$		
	$[11112221121] = (\frac{1}{2}, 0)$				
7,3	$[1111212121]$	$(-1, 3, 1)_{6,3}$			
	$[111112221]$	$(-2, 6, 2)_{6,3}$			
	$[22122221121] = (3, -5)$	$(0, 0, 12)_{6,3}$			
3,7	$[22122212121]$		$(12, 0, 0, 0)_{3,5}$		
	$[22122112221]$		$(4, 5, 0, 0)_{3,5}$		
			$(0, 3, 0, 0)_{3,5}$		
Continued from previous page					

Table C4: Determination of bases of multibrackets in the module  $V^{-\rho}$  on level 0.  
 Note:  $\beta = n_1\beta_1 + n_2\beta_2 - \rho$ . See page 53 for explanation of notation.

$n_1, n_2$	$X_\beta$	$F_1 X_\beta$	$F_2 X_\beta$	$\beta(H_1)$	$\beta(H_2)$
1,0	[1]	$(1)_{0,0}$	0	1	-4
	[2]	0	$(1)_{0,0}$	-4	1
1,1	[21]	$(1)_{0,1}$	$(4)_{1,0}$	-2	-2
	[12]	$(4)_{0,1}$	$(1)_{1,0}$		
2,1	[121]	$(2,1)_{1,1}$	0	0	-5
	[112]	$(0,6)_{1,1}$			
1,2	[221]	0	$(6,0)_{1,1}$	-5	0
	[212]		$(1,2)_{1,1}$		
3,1	[1121]	$(2,1)_{2,1}$	0	2	-8
	[1112]	$(0,6)_{2,1}$			
2,2	[2121]	$(2,1)_{1,2}$	$(5,0)_{2,1}$	-3	-3
	[2112]	$(0,6)_{1,2}$	$(0,5)_{2,1}$		
	[1221]	$(5,0)_{1,2}$	$(6,0)_{2,1}$		
	[1212]	$(0,5)_{1,2}$	$(1,2)_{2,1}$		
1,3	[2221]	0	$(6,0)_{1,2}$	-8	2
	[2212]		$(1,2)_{1,2}$		
4,1	$[11121] = \frac{1}{4}11112$	$(0,1)_{2,1}$	0	4	-11
	[11112]	$(0,4)_{2,1}$			
3,2	[21121]	$(2,1,0,0)_{2,2}$	$(8,0)_{3,1}$	-1	-6
	[21112]	$(0,6,0,0)_{2,2}$	$(0,8)_{3,1}$		
	[12121]	$(3,0,2,1)_{2,2}$	$(5,0)_{3,1}$		
	[12112]	$(0,3,0,6)_{2,2}$	$(0,5)_{3,1}$		
	[11221]	$(0,0,8,0)_{2,2}$	$(6,0)_{3,1}$		
	[11212]	$(0,0,0,8)_{2,2}$	$(1,2)_{3,1}$		

$n_1, n_2$	$X_\beta$	$F_1 X_\beta$	$F_2 X_\beta$	$\beta(H_1)$	$\beta(H_2)$
2,3	[22121]	$(2, 1)_{1,3}$	$(8, 0, 0, 0)_{2,2}$	-6	-1
	[22112]	$(0, 6)_{1,3}$	$(0, 8, 0, 0)_{2,2}$		
	[21221]	$(5, 0)_{1,3}$	$(6, 0, 3, 0)_{2,2}$		
	[21212]	$(0, 5)_{1,3}$	$(1, 2, 0, 3)_{2,2}$		
	[12221]	$(8, 0)_{1,3}$	$(0, 0, 6, 0)_{2,2}$		
	[12212]	$(0, 8)_{1,3}$	$(0, 0, 1, 2)_{2,2}$		
1,4	[22221]	0	$(4, 0)_{1,2}$	-11	4
	$[22212] = \frac{1}{4}[22221]$		$(1, 2)_{1,0}$		
4,2	$[211112] = (0, 4, 0, -6, 0, 4)$	$(0, 4, 0, 0, 0)_{3,2}$	$(11)_{4,1}$	1	-9
	[121121]	$(1, 0, 2, 1, 0, 0)_{3,2}$	$(2)_{4,1}$		
	[121112]	$(0, 1, 0, 6, 0, 0)_{3,2}$	$(8)_{4,1}$		
	[112121]	$(0, 0, 4, 0, 2, 1)_{3,2}$	$(\frac{5}{4})_{4,1}$		
	[112112]	$(0, 0, 0, 4, 0, 6)_{3,2}$	$(5)_{4,1}$		
	[111221]	$(0, 0, 0, 0, 9, 0)_{3,2}$	$(\frac{3}{2})_{4,1}$		
2,4	[111212]	$(0, 0, 0, 0, 0, 9)_{3,2}$	$(\frac{9}{4})_{4,1}$	-9	1
	[222121]	$(\frac{9}{4})_{1,4}$	$(9, 0, 0, 0, 0)_{2,3}$		
	[222112]	$(\frac{3}{2})_{1,4}$	$(0, 9, 0, 0, 0)_{2,3}$		
	[221221]	$(5)_{1,4}$	$(6, 0, 4, 0, 0)_{2,3}$		
	[221212]	$(\frac{5}{4})_{1,4}$	$(1, 2, 0, 4, 0)_{2,3}$		
	[212221]	$(8)_{1,4}$	$(0, 0, 6, 0, 1, 0)_{2,3}$		
	[212212]	$(2)_{1,4}$	$(0, 0, 1, 2, 0, 1)_{2,3}$		
	$[122221] = (4, 0, -6, 0, 4, 0)$	$(11)_{1,4}$	$(0, 0, 0, 4, 0)_{2,3}$		
	[221121]	$(2, 1, 0, 0, 0, 0)_{2,3}$	$(14, 0, 0, 0, 0)_{3,2}$		
	[221112]	$(0, 6, 0, 0, 0, 0)_{2,3}$	$(0, 14, 0, 0, 0)_{3,2}$		
3,3	[212121]	$(3, 0, 2, 1, 0, 0)_{2,3}$	$(5, 0, 6, 0, 0, 0)_{3,2}$	-4	-4
	[212112]	$(0, 3, 0, 6, 0, 0)_{2,3}$	$(0, 5, 0, 6, 0, 0)_{3,2}$		
	[211221]	$(0, 0, 8, 0, 0, 0)_{2,3}$	$(6, 0, 0, 6, 0, 0)_{3,2}$		
	[211212]	$(0, 0, 0, 8, 0, 0)_{2,3}$	$(1, 2, 0, 0, 6)_{3,2}$		
	[122121]	$(6, 0, 0, 2, 1)_{2,3}$	$(0, 0, 8, 0, 0)_{3,2}$		
	[122112]	$(0, 6, 0, 0, 6)_{2,3}$	$(0, 0, 0, 8, 0, 0)_{3,2}$		
	[121221]	$(0, 0, 6, 0, 5, 0)_{2,3}$	$(0, 0, 6, 0, 3, 0)_{3,2}$		
	[121212]	$(0, 0, 0, 6, 0, 5)_{2,3}$	$(0, 0, 1, 2, 0, 3)_{3,2}$		
	[112221]	$(0, 0, 0, 0, 14, 0)_{2,3}$	$(0, 0, 0, 6, 0)_{3,2}$		
	[112212]	$(0, 0, 0, 0, 0, 14)_{2,3}$	$(0, 0, 0, 0, 1, 2)_{3,2}$		
Continued from previous page					

$n_1, n_2$	$X_\beta$	$F_1 X_\beta$	$F_2 X_\beta$	$\beta(H_1)$	$\beta(H_2)$
5,2	$[1121121] = (\frac{2}{3}, \frac{1}{3}, \frac{-1}{6}, \frac{-1}{3})$	$(0, 0, 2, 1, 0, 0)_{3,2}$	0	3	-12
	$[1121112] = (0, \frac{2}{3}, 0, \frac{-4}{3})$	$(0, 0, 0, 6, 0, 0)_{3,2}$			
	$[1112121]$	$(0, 0, 3, 0, 2, 1)_{3,2}$			
	$[1112112]$	$(0, 0, 0, 3, 0, 6)_{3,2}$			
	$[1111221]$	$(0, 0, 0, 0, 8, 0)_{3,2}$			
	$[1111212]$	$(0, 0, 0, 0, 0, 8)_{3,2}$			
4,3	$[2121121]$	$(1, 0, 2, 1, 0, 0, 0, 0, 0, 0)_{3,3}$	$(9, 8, 0, -12, 0, 8)_{4,2}$	-2	-7
	$[2121112]$	$(0, 1, 0, 6, 0, 0, 0, 0, 0, 0)_{3,3}$	$(0, 41, 0, -48, 0, 32)_{4,2}$		
	$[2112121]$	$(0, 0, 4, 0, 2, 1, 0, 0, 0, 0)_{3,3}$	$(0, 5, 9, \frac{-15}{2}, 0, 5)_{4,2}$		
	$[2112112]$	$(0, 0, 0, 4, 0, 6, 0, 0, 0, 0)_{3,3}$	$(0, 0, 20, 0, -21, 0, 20)_{4,2}$		
	$[2111221]$	$(0, 0, 0, 9, 0, 0, 0, 0, 0, 0)_{3,3}$	$(0, 6, 0, -9, 9, 6)_{4,2}$		
	$[2111212]$	$(0, 0, 0, 9, 0, 0, 0, 0, 0, 0)_{3,3}$	$(0, 9, 0, \frac{-27}{2}, 0, 18)_{4,2}$		
	$[1221121]$	$(4, 0, 0, 0, 0, 2, 1, 0, 0, 0)_{3,3}$	$(14, 0, 0, 0, 0, 0)_{4,2}$		
	$[1221112]$	$(0, 4, 0, 0, 0, 0, 6, 0, 0, 0)_{3,3}$	$(0, 14, 0, 0, 0, 0)_{4,2}$		
	$[1212121]$	$(0, 0, 4, 0, 0, 0, 3, 0, 2, 1, 0)_{3,3}$	$(5, 0, 6, 0, 0, 0)_{4,2}$		
	$[1212112]$	$(0, 0, 0, 4, 0, 0, 3, 0, 6, 0, 0)_{3,3}$	$(0, 5, 0, 6, 0, 0)_{4,2}$		
	$[1211221]$	$(0, 0, 0, 4, 0, 0, 0, 8, 0, 0, 0)_{3,3}$	$(6, 0, 0, 0, 6, 0)_{4,2}$		
	$[1211212]$	$(0, 0, 0, 0, 4, 0, 0, 0, 8, 0, 0)_{3,3}$	$(1, 2, 0, 0, 0, 6)_{4,2}$		
	$[1122121]$	$(0, 0, 0, 0, 0, 10, 0, 0, 0, 2, 1)_{3,3}$	$(0, 0, 8, 0, 0, 0)_{4,2}$		
	$[1122112]$	$(0, 0, 0, 0, 0, 0, 10, 0, 0, 0, 6)_{3,3}$	$(0, 0, 0, 8, 0, 0)_{4,2}$		
	$[1121221]$	$(0, 0, 0, 0, 0, 0, 0, 10, 0, 5, 0)_{3,3}$	$(0, 0, 6, 0, 3, 0)_{4,2}$		
	$[1121212]$	$(0, 0, 0, 0, 0, 0, 0, 0, 10, 0, 5)_{3,3}$	$(0, 0, 1, 2, 0, 3)_{4,2}$		
	$[1112221]$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 18, 0)_{3,3}$	$(0, 0, 0, 0, 6, 0)_{4,2}$		
	$[1112212]$	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 18)_{3,3}$	$(0, 0, 0, 0, 1, 2)_{4,2}$		
5,3	$[21112121]$	$(0, 0, 3, 0, 2, 1, 0, 0, 0, 0, 0, 0, 0, 0)_{4,3}$	$(12, 0, 0, 0)_{5,2}$	0	-10
	$[21112112]$	$(0, 0, 0, 3, 0, 6, 0, 0, 0, 0, 0, 0, 0, 0)_{4,3}$	$(0, 12, 0, 0)_{5,2}$		
	$[21111221]$	$(0, 0, 0, 0, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0)_{4,3}$	$(0, 0, 12, 0)_{5,2}$		
	$[21111212]$	$(0, 0, 0, 0, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0)_{4,3}$	$(0, 0, 0, 12)_{5,2}$		
	$[12121121]$	$(2, 0, 0, 0, 0, 1, 0, 2, 1, 0, 0, 0, 0, 0, 0)_{4,3}$	$(6, 7, \frac{-3}{2}, -7)_{5,2}$		
	$[12121112]$	$(0, 2, 0, 0, 0, 0, 1, 0, 6, 0, 0, 0, 0, 0, 0)_{4,3}$	$(0, 34, 0, \frac{-59}{2})_{5,2}$		
	$[12112121]$	$(0, 0, 2, 0, 0, 0, 0, 4, 0, 2, 1, 0, 0, 0, 0, 0)_{4,3}$	$(9, \frac{5}{2}, 0, \frac{-5}{2})_{5,2}$		
	$[12112112]$	$(0, 0, 0, 2, 0, 0, 0, 0, 4, 0, 6, 0, 0, 0, 0, 0)_{4,3}$	$(0, 19, 0, -10)_{5,2}$		
	$[12111221]$	$(0, 0, 0, 0, 2, 0, 0, 0, 0, 9, 0, 0, 0, 0, 0, 0)_{4,3}$	$(0, 3, 9, -3)_{5,2}$		
	Continued from previous page				

$n_1, n_2$	$X_\beta$	$F_2 X_\beta$	$F_1 X_\beta$	$F_2 X_\beta$	$\beta(H_1)$	$\beta(H_2)$
	[12111212]		(0,0,0,0,0,2,0,0,0,0,0,0,0,0,0,0) <sub>4,3</sub>	$(0, \frac{9}{2}, 0, \frac{9}{2})_{5,2}$	0	-10
	[11221121]		(0,0,0,0,0,6,0,0,0,0,0,2,1,0,0,0) <sub>4,3</sub>	$(\frac{28}{3}, \frac{14}{3}, \frac{-7}{3}, \frac{-14}{3})_{5,2}$		
	[11221112]		(0,0,0,0,0,0,6,0,0,0,0,0,6,0,0,0) <sub>4,3</sub>	$(0, 28, 0, -21)_{5,2}$		
	[11212121]		(0,0,0,0,0,0,0,6,0,0,0,3,0,2,1,0,0) <sub>4,3</sub>	$(\frac{28}{3}, \frac{5}{3}, \frac{6}{3}, \frac{-5}{3})_{5,2}$		
	[11212112]		(0,0,0,0,0,0,0,0,6,0,0,0,3,0,6,0,0) <sub>4,3</sub>	$(0, 16, 0, \frac{-15}{2})_{5,2}$		
	[11211221]		(0,0,0,0,0,0,0,0,0,6,0,0,0,8,0,0,0) <sub>4,3</sub>	$(4, 2, 5, -2)_{5,2}$		
	[11211212]	5,3	(0,0,0,0,0,0,0,0,0,0,6,0,0,0,8,0,0) <sub>4,3</sub>	$(\frac{2}{3}, \frac{13}{3}, \frac{-1}{6}, \frac{8}{3})_{5,2}$		
	[11122121]		(0,0,0,0,0,0,0,0,0,0,0,12,0,0,0,2,1) <sub>4,3</sub>	$(8, 0, 0, 0)_{5,2}$		
	[11122112]		(0,0,0,0,0,0,0,0,0,0,0,0,12,0,0,0,6) <sub>4,3</sub>	$(0, 8, 0, 0)_{5,2}$		
	[11121221]		(0,0,0,0,0,0,0,0,0,0,0,0,0,12,0,5,0) <sub>4,3</sub>	$(6, 0, 3, 0)_{5,2}$		
	[11121212]		(0,0,0,0,0,0,0,0,0,0,0,0,0,0,12,0,5) <sub>4,3</sub>	$(1, 2, 0, 3)_{5,2}$		
	[11112221] =		(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,20,0) <sub>4,3</sub>	$(0, 0, 6, 0)_{5,2}$		
	(0,0,-1,0,0,0,0,4,0,0,0,0,0,-6,0,0,0,4,0)		(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,20) <sub>4,3</sub>	$(0, 0, 1, 2)_{5,2}$		
	[11112212] =					
(0,0,0,-1,0,0,0,0,4,0,0,0,0,0,-6,0,0,0,4)						
Continued from previous page						

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